Homework #6, EECS 206, Fall 2002. Due Fri. Oct. 25, by 4:30PM

Notes

- Review the HW policies on HW1!
- Reminder: Quiz 2 will (finally!) be Oct. 22-24. It covers Labs 2 and 3.
- Reading:
  - "Part 5" lecture notes.
  - "Note: The $\Delta^2/12$ formula" at end of course pack (p126).
  - "Notes: The Distortion of Transform Coding" at end of course pack (p128).
  - Text chapter 4.

Skills and Concepts

- spectra and applications
- A/D and D/A conversion: sampling, quantization, interpolation

Problems

1. [20] The amplitude modulation system described in lecture and in the lecture notes is called "double sideband (DSB) modulation," and is the standard for commercial AM radio. An alternative approach is "single sideband (SSB) modulation" and can be described by the following block diagram:

$$\text{Audio: } v(t) \to \underset{\substack{\uparrow \\ d}}{\bigoplus} \longrightarrow \underset{\substack{\uparrow \\ \cos(2\pi f_c t)}}{\bigotimes} \to \boxed{\text{High-pass filter}} \to x(t) \to \text{Antenna}$$

In this system, the "filter" is a circuit that removes all signal components having frequencies that are less than the carrier frequency f_c and greater than $-f_c$. The filter leaves all other frequency components untouched. The purpose of this problem is to reinforce your understanding of spectra and the AM radio application in particular, and to explore the advantages and disadvantages of this alternative modulation scheme.

(a) [10] Suppose that $v(t) = A\cos(2\pi f_v t)$, where $A < d$ and $f_v \ll f_c$. Sketch the spectrum of $x(t)$.
Hint: first sketch the spectrum of the signal that is the *input* to the filter.

(b) [5] Suppose the maximum audio frequency f_v of interest is just a bit less than 5 kHz. If SSB were used by multiple radio stations and you wanted to avoid signal interference, what would be the minimum spacing of the carrier frequencies of the different stations? (Explain.) Hint: the best answer involves a number that is smaller than 10kHz.

(c) [0] In light of the preceding answer, why do you think the double sideband method was chosen (long ago) as the standard for commercial AM radio?
Single sideband *is* used in a variety of communications applications, including NASA's link with the space shuttle: `http://www.teachspace.org/resources/shortwave.html` .

(d) [5] If $f_v = 5$kHz and $f_c = 100$kHz and we want to sample $x(t)$ and be able to later recover $x(t)$ from those samples, what inequality must the sampling frequency f_s satisfy?

2. [15] For each of the following discrete-time signals, determine the frequency. Also determine whether the signal is periodic. If so, determine its fundamental period; if not, sketch its two-sided spectrum.

(a) [5] $x[n] = \cos\left(\frac{\pi}{5}n + \pi/3\right)$

(b) [5] $x[n] = \cos\left(\frac{7\pi}{8}n - \pi/4\right)$

(c) [5] $x[n] = \cos(5n + \pi/7)$

(d) [0] Is there a simple relationship between the frequency and the period? How do you explain the answers to (a) and (b) considering that the frequency in (b) is higher than the frequency in (a)?

Hint: you may want to use MATLAB to plot these signals. However, show *analytically* how you determined the frequencies and periods (not just by looking at the graph).

3. [30] In lecture, we demonstrated that one could perform audio signal compression by computing the DFT of a signal, discarding the "small" DFT coefficients, and then reconstructing (a close approximation to) the original signal using the inverse DFT (the synthesis formula). This is the essence of how MP3 digital audio compression and JPEG digital image compression work. The purpose of this problem is to give you firsthand experience with these signal compression methods.
   (a) [0] Download the MATLAB file `dft_chop.m` from the homework page of the course web site and run it. Do this right away to avoid last minute web server / MATLAB problems!
   (b) [10] Modify the program so that it computes and prints the mean squared error (MSE) between the original signal $x[n]$ and the signal $y[n]$ that is reconstructed from the "large" DFT coefficients.
   (c) [0] The program includes a user-selectable "threshold" that determines what is a "small" DFT coefficient that should be discarded. Explore various values for this threshold, and observe the effect on the MSE.
   (d) [20] Determine the largest possible threshold that satisfies the accuracy design criterion $\text{MSE}(x, y) < 0.0020$. You can do this by trial and error (to a modest number of decimal places), or you may write a loop to do this "automatically." (Either way, include a printout of your modified program.)
   (e) [0] For the threshold you found, how much is the data storage reduced?

4. [30] An engineer is working with the 5-periodic signal $x(t)$ whose Fourier series coefficients are given by $\alpha_k = 3^{-|k|},\ k \in \mathbb{Z}$. She needs to plot this signal, but of course cannot do computation with an infinite number of coefficients. Instead, she synthesizes an *approximation* to $x(t)$ by using $2K + 1$ coefficients as follows:

$$\hat{x}_K(t) = \sum_{k=-K}^{K} \alpha_k e^{\jmath (2\pi/T)kt}.$$

   To save computation, she would like to use the smallest value of K possible while maintaining adequate accuracy for her problem. Specifically, she wants the normalized RMS error to be less than 0.1%. In other words, she wants the smallest K for which $\text{NRMS}(\hat{x}_K, x) = \dfrac{\text{RMS}(\hat{x}_K - x)}{\text{RMS}(x)} < 0.001$.
   The purpose of this problem is to provide and example of how spectra concepts arise in a *design* problem. Unlike many "number crunching" problems, design problems usually involve multiple steps. Here are the steps for this situation.
   (a) [5] Sketch the spectrum of $x(t)$.
   (b) [5] Sketch the spectrum of $\hat{x}_K(t)$ for $K = 2$.
   (c) [5] Sketch the spectrum of $\hat{x}_K(t) - x(t)$ for a "generic" value of $K \geq 1$.
   (d) [5] Show that $\text{RMS}(x) = \sqrt{5}/2$. Hint: use Parseval and a geometric series formula.
   (e) [5] Show that

$$\text{RMS}(\hat{x}_K - x) = \frac{1}{2 \cdot 3^K}.$$

   (f) [5] Find the smallest K that satisfies the error criterion.

5. [10] Consider the 8-periodic discrete-time sawtooth signal $x[n]$ described by $x[n] = \begin{cases} 10n, & n = 0, \ldots, 7 \\ \text{periodic}, & \text{otherwise.} \end{cases}$
   One can show that the 8-point DFT of this signal is given by $X[k] = \dfrac{10}{e^{-\jmath(\pi/4)k} - 1}$.
   Without using the DFT synthesis formula or MATLAB, sketch (for $n = 0, \ldots, 7$) the signal $y[n]$ whose 8-point DFT is given by $Y[k] = \dfrac{5e^{-\jmath(\pi/2)k}}{e^{-\jmath(\pi/4)k} - 1}$.

6. [15] The purpose of this problem is to compare *quantitatively* the two simplest signal interpolation methods that are used in D/A conversion: zero-order hold, and linear interpolation. Consider the following scenario:

$$x(t) \to \boxed{\text{C-to-D}} \to x[n] \to \boxed{\text{D-to-C}} \to \hat{x}(t).$$

A good interpolator should have $\hat{x}(t) \approx x(t)$, so MSD$(\hat{x} - x)$ should be small. In this problem we consider the 16-periodic pulse-train signal $x(t) = \begin{cases} 1, & 0 \le t < 2 \\ 0, & 2 \le t < 14 \\ 1, & 14 \le t < 16. \end{cases}$

(a) [5] Assume the sampling interval is $T_s = 4$. Sketch the samples $x[n]$ for $n = 0, 1, \ldots, 8$.

(b) [5] Let $y(t)$ denote the result of zero-order hold signal interpolation from $x[n]$. Sketch both $x(t)$ and $y(t)$ on the same graph for $0 \le t \le 32$. (Use two colors.) Does $y(t)$ look like a good approximation to $x(t)$?

(c) [5] Let $z(t)$ denote the result of linear interpolation from $x[n]$. Sketch both $x(t)$ and $z(t)$ on the same graph for $0 \le t \le 40$. Does $z(t)$ look like a better or worse approximation to $x(t)$ than $y(t)$ did?

(d) [0] Show that MSD$(x, z) = 1/24$ and determine MSD(x, y) using time-domain calculations. Quantitatively, for the signal $x(t)$, which interpolation method is better and why?

(e) [0] Suppose that instead we used the ideal sinc interpolator: $\hat{x}(t) = \sum_n x[n]\, p(t - nT_s)$, where $p(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s}$. Would MSD$(x, \hat{x}) = 0$, *i.e.*, would $\hat{x} = x$ in this case? Why or why not?

7. [15] The purpose of this problem is to illustrate aliasing of sinusoidal signals.
A continuous-time sinusoidal signal is sampled at the sampling rate $f_s = 2000$Hz, yielding the following discrete-time signal:

$$x[n] = 10 \cos\left(\frac{7}{8}\pi n + \frac{\pi}{3}\right).$$

Give *three* examples of continuous-time signals that could have been the original signal:

(a) [5] $x_0(t)$ corresponding to an unaliased case,

(b) [5] $x_1(t)$ corresponding to "case 1" aliasing (see lecture notes), and

(c) [5] $x_2(t)$ corresponding to "case 2" aliasing (folding).

Optional Problems

8. [10] (Optional extra credit challenge problem. No help will be given in office hours for such problems.)
Back in the 20th century, engineers used mathematical reference books containing useful formulas such as

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}, \qquad \text{and} \qquad \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{8}.$$

In the 21st century, one can find such formulas at web sites like `http://mathworld.wolfram.com`. Have you ever wondered how such formulas were discovered in the first place? Interestingly, all of the above formulas can be derived using concepts covered in EECS 206 so far. For extra credit, derive any of the formulas above using concepts from EECS 206. (If you know of non-206 approaches to these problems, please tell Prof. Fessler!)

9. [10] (Optional extra credit challenge problem. No help will be given in office hours for such problems.)
Show that the N-point DFT of the discrete-time ramp signal $x[n] = n$ is given by

$$X[k] = \begin{cases} N(N-1)/2, & k = 0 \\ \left[e^{-\jmath \frac{2\pi}{N} k} - 1\right]^{-1}, & k \neq 0. \end{cases}$$