Approximating signal characteristics using Riemann sums

This note supplements some formulas given in the Lab 1 background material without derivation.

Recall from the development of the **Riemann integral** in calculus that if N is large, then

$$\int_a^b f(t)\,\mathrm{d}t \approx \frac{b-a}{N} \sum_{k=1}^N f(t_k)\,,$$

where the interval $[a, b]$ is partioned into N segments with left endpoints given by $t_k = a + \frac{k-1}{N}(b-a)$.

We can use this approximation to compute signal characteristics like mean and energy of continuous-time signals by first **sampling** those signals and then using the discrete-time formulas for signal characteristics, *with a slight modification* in some cases.

Consider first the problem of calculating the **energy** of a continuous-time signal $x(t)$ from its samples $\{x[n]\}$ defined by

$$x[n] = x(nT_\mathrm{s})\,,$$

where T_s denotes the sampling rate.

Further assume that $t_1 = n_1 T_\mathrm{s}$ and $t_2 = (n_2+1)T_\mathrm{s}$ for some integers n_1 and n_2, so that $N = n_2 - n_1 + 1$ is the total number of samples, and $t_2 - t_1 = NT_\mathrm{s}$.

Applying the Riemann approximation:

$$\begin{aligned} E(x) &= \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t \approx \frac{t_2 - t_1}{N} \sum_{k=1}^N x^2\left(t_1 + \frac{k-1}{N}(t_2 - t_1)\right) \\ &= T_\mathrm{s} \sum_{k=1}^N x^2(n_1 T_\mathrm{s} + (k-1)T_\mathrm{s}) = T_\mathrm{s} \sum_{n=n_1}^{n_2} x^2[n]. \end{aligned}$$

In summary, we can approximate the energy of a sampled signal as follows:

$$\boxed{E(x) = \int_{t_1}^{t_2} x^2(t)\,\mathrm{d}t \approx T_\mathrm{s} \sum_{n=n_1}^{n_2} x^2[n] = \texttt{Ts * mean(x.\^{}2)}.}$$

Notice how the "extra" factor T_s comes out front due to the "width of the rectangle" in the Riemann approximation.

Now consider instead the **mean** signal value:

Applying the Riemann approximation:

$$\begin{aligned} M(x) &= \frac{1}{t_2 - t_1} \cdot \int_{t_1}^{t_2} x(t)\ \mathrm{d}t \approx \frac{1}{t_2 - t_1} \cdot \frac{t_2 - t_1}{N} \sum_{k=1}^N x\left(t_1 + \frac{k-1}{N}(t_2 - t_1)\right) \\ &= \frac{1}{N} \sum_{k=1}^N x(n_1 T_\mathrm{s} + (k-1)T_\mathrm{s}) = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} x[n]\,. \end{aligned}$$

In summary

$$\boxed{M(x) = \int_{t_1}^{t_2} x(t)\ \mathrm{d}t \approx \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} x[n] = \texttt{mean(x)}.}$$

Notice how this time there is no "extra" factor because it cancels out due to the **normalization** by the signal duration.