Supplemental Lecture Notes for Ch. 2

Outline for Ch. 2
- Convert between 3 representations of sinusoid
  - formula $x(t) = Acos(\omega_0 t + \phi)$
  - amplitude, frequency (or period), phase
  - graph/plot
- Simplify sums of sinusoids *of same frequency*
  - trigonometry
  - phasors
- Complex arithmetic
  - Euler's identities
  - addition/subtraction
  - cartesian / polar / complex exponential form
  - multiplication / division
  - polynomial roots
- Complex exponential signals
- Beat frequencies

Overview of sinusoids

Why?
- Occur in nature
  - tuning fork
  - flute
  - solution to many differential equations
- Engineering systems
  - power generation (rotating equipment)
  - laser
  - oscillator (modulators for comm)
- LTI systems
  - sinusoid in makes sin out: unique!
  - motivates considering other signals as sums of sinusoids

Demo: filterdemoc2.m

Demo: show cos( ) + cos( ) examples

Same amplitude sum of cos() is a ”special case”
Need better method...

Demos
- rotating phasor: cos is real, sin is imag
- helix for complex exponential
- cos from sum of two rotating phasors in opposite directions

Terminology

The book calls $r\angle\theta$ the **polar form** for a complex number, and calls $r e^{\jmath\theta}$ the **complex exponential form**. My view is that these forms are so similar, since they both involve the magnitude and the angle of the complex value, that it is acceptable to refer to $r e^{\jmath\theta}$ as the "polar form" since it is easier to say (and type) than "complex exponential form."

Complex division

How do we compute $z_3 = z_1/z_2$ with complex numbers?

- **Polar form**

$$\frac{z_1}{z_2} = \frac{r_1 e^{\jmath\theta_1}}{r_2 e^{\jmath\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{\jmath(\theta_1-\theta_2)}$$

- **Cartesian form** (complex conjugate of denominator simplifies this to a multiplication problem)

$$\frac{z_1}{z_2} = \frac{z_1 z_2^\star}{z_2 z_2^\star} = \frac{z_1 z_2^\star}{|z_2|^2} = \frac{x_1+\jmath y_1}{x_2+\jmath y_2}\cdot\frac{x_2-\jmath y_2}{x_2-\jmath y_2} = \frac{(x_1x_2+y_1y_2)+\jmath(x_2y_1-x_1y_2)}{x_2^2+y_2^2} = \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)+\jmath\left(\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}\right)$$

Example. Find z_1/z_2 when $z_1 = 16e^{-\jmath 2\pi/3}$ and $z_2 = 4-\jmath 4\sqrt{3}$.

- Cartesian solution desired
  $z_1 = 16e^{-\jmath 2\pi/3} = 16(-1/2-\jmath\sqrt{3}/2) = -8-\jmath 8\sqrt{3}$

$$\frac{z_1}{z_2} = \frac{-8-\jmath 8\sqrt{3}}{4-\jmath 4\sqrt{3}} = \frac{(-8-\jmath 8\sqrt{3})(4+\jmath 4\sqrt{3})}{4^2+(4\sqrt{3})^2} = \frac{(-32+32\cdot 3)+\jmath(-32-32)\sqrt{3})}{64} = 1-\jmath\sqrt{3}$$

- Polar solution desired
  $z_2 = 4-\jmath 4\sqrt{3} = \sqrt{4^2+4^2\cdot 3}\, e^{-\jmath\pi/3} = 8e^{-\jmath\pi/3}$

$$\frac{z_1}{z_2} = \frac{16e^{-\jmath 2\pi/3}}{8e^{-\jmath\pi/3}} = 2e^{-\jmath\pi/3}$$

Sanity check: $2e^{-\jmath\pi/3} = 2[\cos(-\pi/3)+\jmath\sin(-\pi/3)] = 2[1/2-\jmath\sqrt{3}/2] = 1-\jmath\sqrt{3}$, so the two answers indeed agree.

Complex roots

The roots of the polynomial $z^2+1=0$ are $z=\pm\jmath=\pm\sqrt{-1}$. This is a second-degree polynomial so it has two roots.
One might say that $\jmath$ was "invented" so that the **fundamental theorem of algebra** works: an nth degree polynomial has n roots.

What are the roots of the polynomial $z^3+8=0$? It is a third-degree polynomial so it has three roots.
An equivalent question would be: determine $(-8)^{1/3}$.

Do we need to invent a $\sqrt[3]{-1}$ to solve this problem? Fortunately, no!

Strategy: solve $z^3=-8$ by using polar form, $z=re^{\jmath\theta}$.
So $r^3 e^{\jmath 3\theta} = -8 = 8e^{\jmath\pi}$.
Equating the *magnitudes*, we see that $r^3=8$ and since r is real, we have $r=2$. That's the easy part.
Fact: if $e^{\jmath\phi}=e^{\jmath\gamma}$, then $\phi=\gamma+k2\pi$ for some integer k.
Equating the *phases*, we see $3\theta=\pi+k2\pi$ so $\theta=\frac{\pi}{3}+k\frac{2\pi}{3}$. ***Picture***
Choosing three consecutive integers $k=-1,0,1$, we have $\theta\in\{\pm\pi/3,\pi\}$
So the roots are $z=2e^{\jmath\pi}=-2$ and $z=2e^{\jmath\pm 2\pi/3}=1\pm\jmath\sqrt{3}$.

Caution: MATLAB's `-8 ^ (1/3)` only gives one of the three possible values!

Caution: $\boxed{(e^{\jmath\theta})^n = e^{\jmath n\theta} \text{ when } n\in\mathbb{Z}}$ (integers). But $\boxed{(e^{\jmath\theta})^{1/n} = e^{\jmath(\theta/n+k2\pi/n)} \text{ for } k\in\mathbb{Z}.}$

More practice

Use the `zdrill` mfile in DSP First toolbox for practice!

Sums of sinusoidal signals of *same frequency* (This is a primary motivation for complex numbers!)

Example. Find the **amplitude** A and the **phase** θ of the following sum-of-sinusoids signal:

$$x(t) = 2\cos(5t + \pi/4) + 2\sqrt{2}\sin(5t) \stackrel{?}{=} A\cos(5t + \theta).$$

Note that the **frequency** remains unchanged!

Most important formula: $\boxed{\cos(\phi) = \mathrm{Re}\left(e^{\jmath\phi}\right).}$ Also recall that $\sin x = \cos\left(x - \frac{\pi}{2}\right)$.

$$\begin{aligned}
x(t) &= 2\cos(5t + \pi/4) + 2\sqrt{2}\cos(5t - \pi/2) \\
&= \mathrm{Re}\left(2e^{\jmath(5t+\pi/4)} + 2\sqrt{2}e^{\jmath(5t-\pi/2)}\right) \\
&= \mathrm{Re}\left(e^{\jmath 5t}\left(2e^{\jmath\pi/4} + 2\sqrt{2}e^{-\jmath\pi/2}\right)\right) \\
&= \mathrm{Re}\left(e^{\jmath 5t}\left(2(\sqrt{2}/2 + \jmath\sqrt{2}/2) + 2\sqrt{2}(-\jmath)\right)\right) \\
&= \mathrm{Re}\left(e^{\jmath 5t}\left(\sqrt{2} - \jmath\sqrt{2}\right)\right) \\
&= \mathrm{Re}\left(e^{\jmath 5t} 2e^{-\jmath\pi/4}\right) = \mathrm{Re}\left(2e^{\jmath(5t-\pi/4)}\right) = 2\cos(5t - \pi/4)
\end{aligned}$$

The complex values first appear in polar form, yet we must add them so cartesian form is more convenient. Then the final form requires polar form again.

This example was "cooked" for chalkboard use without a calculator.
In practice, these problems are solved easily using any scientific calculator that handles complex numbers in polar form.
You need such a calculator for the exams!

General rule for summing sinusoidal signals of the same frequency:

$$x(t) = \sum_k A_k\cos(\omega_0 t + \theta_k) = A\cos(\omega_0 t + \theta), \quad \text{where } \boxed{Ae^{\jmath\theta} = \sum_k A_k e^{\jmath\theta_k}.}$$

Note that all that really enters into the calculation is the sum of the terms of the form $A_k e^{\jmath\theta_k}$. These terms are called **phasors**, particularly in the context of electrical circuits. This representation simplifies calculations with resistors, capacitors, and inductors (RLC circuits) since one can solve many problems (for sinusoidal signals) using the phasors and the (complex) impedance of each circuit element.

Summary: the key step in this approach was writing

$$x_1(t) = A_1\cos(\omega_0 t + \phi_1) = \mathrm{Re}\left(A_1 e^{\jmath(\omega_0 t + \phi_1)}\right).$$

A (complex) signal of the form $\bar{x}_1(t) = A_1 e^{\jmath(\omega_0 t+\phi_1)}$ is called a **complex exponential signal**.
Another name for it is a **rotating phasor**.

What about a signal of the form $x(t) = \exp(-2t)$? This is an ordinary **exponential signal**; it is not "complex."

Representing **sinusoidal signals** as the real part of **complex exponential signals** allows us to **add** such signals "easily" using **complex arithmetic** rather than **trigonometry**.

Relationship between sinusoidal signals and complex exponential signals

- Viewpoint 1:

$$x(t) = A\cos(\omega_0 t + \phi) = \mathrm{Re}\Big(A e^{\jmath(\omega_0 t+\phi)}\Big) = \mathrm{Re}\big(\big(A e^{\jmath\phi}\big)\, e^{\jmath\omega_0 t}\big)\,, \text{ where } \big(A e^{\jmath\phi}\big) \text{ is the } \textbf{phasor}\,.$$

- Viewpoint 2:

$$\begin{aligned} x(t) = A\cos(\omega_0 t + \phi) &= \frac{A}{2} e^{\jmath(\omega_0 t+\phi)} + \frac{A}{2} e^{-\jmath(\omega_0 t+\phi)} \\ &= \frac{1}{2}\big(A e^{\jmath\phi}\big)\, e^{\jmath\omega_0 t} + \frac{1}{2}\big(A e^{-\jmath\phi}\big)\, e^{-\jmath\omega_0 t}. \end{aligned}$$

  Note that the phasor and its complex conjugate appear!
  So a sinusoidal signal is the sum of two **rotating phasors**.
  Why? Because of **inverse Euler** identity: $\cos\theta = \frac{1}{2}e^{\jmath\theta} + \frac{1}{2}e^{-\jmath\theta}$.

Note that there is a **negative frequency** for the second complex exponential.
This corresponds to a **rotating phasor** that has **clockwise** rotation in the complex plane.
We need the combination of the two rotating phasors having opposite directions of rotation so that when added together, the imaginary parts cancel out and we are left with the real part which is the cosine part.
We never need a negative frequency for sinusoidal signals, only for complex exponential signals.

Plotting complex exponential signals

There are three ways to plot a complex exponential signal.

$$\bar{x}(t) = A e^{\jmath(\omega_0 t+\phi)} = A\cos(\omega_0 t + \phi) + \jmath\, A\sin(\omega_0 t + \phi) = \mathrm{Re}(\bar{x}(t)) + \jmath\,\mathrm{Im}(\bar{x}(t))\,.$$

1. Separate plots of real and imaginary parts
   (Picture) of two sinusoids
2. Plot in complex plane (**rotating phasor**)
   ◦ magnitude $|\bar{x}(t)| = A$
   ◦ angle $\angle\bar{x}(t) = \omega_0 t + \phi$

   (Picture) of counter-clockwise rotation (for positive ω_0)
3. 3D plot: real and imaginary vs time ***(Picture)*** of **helix**

Complex signals

We began the course defining simple signal characteristics and simple signal operations. Those definitions were for **real signals**, although many apply to **complex signals** too.

A complex signal has a **real part** and an **imaginary part** as follows:

$$z(t) = x(t) + iy(t).$$

Most signal characteristics are easy generalizations of those defined for real signals. and are described at end of Part 1 lecture notes.

Example. Mean of complex signal.

$$\begin{aligned} M(z) &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} z(t) \, \mathrm{d}t = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [x(t) + \jmath\, y(t)] \, \mathrm{d}t \\ &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x(t) \, \mathrm{d}t + \jmath \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} y(t) \, \mathrm{d}t = M(x) + \jmath\, M(y). \end{aligned}$$

An important difference is that for complex signal properties, anywhere we had the **squared value** $x^2(t)$ before, we replace it with the **magnitude squared** $|z(t)|^2 = z(t) z^\star(t) = x^2(t) + y^2(t)$.

Example. The **energy** of a complex signal $z(t)$ is $\boxed{E(z) = \int_{t_1}^{t_2} |z(t)|^2 \, \mathrm{d}t.}$

Another related difference is that we define correlation for complex signals as follows:

$$C(z_1, z_2) = \int_{t_1}^{t_2} z_1(t) z_2^\star(t) \, \mathrm{d}t.$$

One reason for this choice is that it satisfies $E(z) = C(z, z)$.

The **signal operations** like time scaling, time shift, etc. all apply to both the real part and the imaginary part.

Similar considerations for **discrete-time** signals.

Example. Find the energy of the signal $z(t) = \begin{cases} \mathrm{e}^{\jmath 5t} \mathrm{e}^{-2t}, & t > 0 \\ 0, & \text{otherwise.} \end{cases}$

$$E(z) = \int_0^\infty |z(t)|^2 \, \mathrm{d}t = \int_0^\infty \left|\mathrm{e}^{\jmath 5t}\right|^2 \left|\mathrm{e}^{-2t}\right|^2 \, \mathrm{d}t = \int_0^\infty \mathrm{e}^{-4t} \, \mathrm{d}t = \frac{1}{4}.$$

Beat frequencies (Ch. 3.2)

Are complex exponential signals useful for summing sinusoidal signals with *different* frequencies? Sometimes!

Ch. 3 on **spectra** is all about sinusoids of different frequencies!

Example. Sum of two "nearly same" frequencies. (Same amplitude for simplicity, not necessity.)

$$x(t) = A\cos(\omega_0 t) + A\cos(\omega_1 t)$$

where $\omega_1 - \omega_0$ is "small."

Define the center frequency $\bar{\omega} = \frac{\omega_0+\omega_1}{2}$ and $\Delta = \omega_1 - \bar{\omega} = \bar{\omega} - \omega_0$ for $\omega_1 > \omega_0$.

Assumption: $\Delta \ll \bar{\omega}$.

(Picture) .

This type of $x(t)$ has a notable auditory property. Can we describe it mathematically?

$$\begin{aligned}
x(t) &= \mathrm{Re}\left(A e^{\jmath\omega_0 t} + A e^{\jmath\omega_1 t}\right) \\
&= A\mathrm{Re}\left(e^{\jmath(\bar{\omega}-\Delta)t} + e^{\jmath(\bar{\omega}+\Delta)t}\right) \\
&= A\mathrm{Re}\left(e^{\jmath\bar{\omega}t}\left(e^{-\jmath\Delta t} + e^{\jmath\Delta t}\right)\right) \\
&= A\mathrm{Re}\left(e^{\jmath\bar{\omega}t} 2\cos(\Delta t)\right) \\
&= 2A\cos(\Delta t)\mathrm{Re}\left(e^{\jmath\bar{\omega}t}\right) = 2A\cos(\Delta t)\cos(\bar{\omega}t).
\end{aligned}$$

So we have the product of a slowly changing sinusoidal signal times a higher frequency sinusoidal signal.

(Picture) of signals and their product.

Demo of closely spaced case and harmonically-related case.

Also try summing square wave and triangular wave.
Why similar? Sum of sinusoids!

Sinusoids? Enough already!

Yes, the real world has many signals that are far more interesting than sinusoidal signals.

Joseph Fourier showed in 1807 that most any signal can be expressed as the sum of (a lot of) sinusoidal signals (not of the same frequency though!), simply by carefully choosing the frequencies, amplitudes, and phases.

"Joe" did it almost 200 years ago without calculators or MATLAB...