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**Computing the DFT**

There are three basic methods for "manually" determining the DFT of a signal:

- matching the DFT coefficients "by inspection,"
- using the DFT analysis formula, or
- combining the above with DFT properties.

The same techniques also work for the inverse DFT since it is almost the same formula!

Recall the $N$-point DFT formulas:

$$\text{Analysis: } X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-\jmath(2\pi/N)kn} \qquad \text{Synthesis: } x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\jmath(2\pi/N)kn}.$$

Example. **Method 1: coefficient matching**

Determine the 32-point DFT the following signal $x[n] = 20\sin^2(3\pi/8n)$. The purpose of the $N$-point DFT is to express the signal as a sum of $N$ complex exponentials:

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\jmath(2\pi/N)kn} = X[0] + X[1]\, e^{\jmath(2\pi/N)n} + X[2]\, e^{\jmath(2\pi/N)2n} + \cdots + +X[N-1]\, e^{\jmath(2\pi/N)(N-1)n}. \qquad \text{(4d-1)}$$

If we can find such an expression directly, then we do not need to use the analysis formula.
In this case, apply an inverse Euler identity:

$$\begin{aligned} x[n] &= 20\sin^2(3\pi/8n) = 20\left(\frac{e^{\jmath 3\pi/8n} - e^{-\jmath 3\pi/8n}}{2\jmath}\right)^2 \\ &= 20\left(\frac{e^{\jmath(2\pi/32)6n} - e^{-\jmath(2\pi/32)6n}}{2\jmath}\right)^2 = 5\left(2 - e^{\jmath(2\pi/32)12n} - e^{-\jmath(2\pi/32)12n}\right) \\ &= 10 - 5e^{\jmath(2\pi/32)12n} - 5e^{-\jmath(2\pi/32)12n} = 10 - 5e^{\jmath(2\pi/32)12n} - 5e^{\jmath(2\pi/32)20n}, \end{aligned}$$

where in the last line we used the $2\pi$ periodicity of $e^{\jmath t}$, adding $2\pi n$ to the exponent. Considering the final form, comparing to (4d-1) we see that the 32-point DFT of $x[n]$ is given by:

$$X[k] = \begin{cases} 10, & k = 0 \\ -5, & k = 12 \\ -5, & k = 20 \\ 0, & \text{otherwise.} \end{cases}$$

We see that this signal has a DC term and two other complex exponential terms.

Example. **Method 2: analysis formula**

Determine the 32-point DFT the following signal $y[n] = (1/5)^n$. Note that this is an infinite duration signal, so we are only computing the DFT of a segment of it. We chose $N = 32$ here simply for illustration.

Substitute $y[n]$ into the analysis formula and use a geometric series formula to help simplify:

$$\begin{aligned} Y[k] &= \frac{1}{32}\sum_{n=0}^{31}(1/5)^n\, e^{-\jmath(2\pi/32)kn} = \frac{1}{32}\sum_{n=0}^{31}\left((1/5)\, e^{-\jmath(2\pi/32)k}\right)^n \\ &= \frac{1}{32}\frac{1-\left((1/5)\, e^{-\jmath(2\pi/32)k}\right)^{32}}{1-(1/5)\, e^{-\jmath(2\pi/32)k}} = \frac{1}{32}\frac{1-1/5^{32}}{1-(1/5)\, e^{-\jmath(2\pi/32)k}}. \end{aligned}$$

A plot of the magnitude spectrum shows that this signal has nearly the same power at all frequencies, with a bit more at the lower frequencies.

Example. **Method 3: using properties**

Determine the 32-point DFT the following signal $z[n] = 40\sin^2(3\pi/8(n-4)) + 7(1/5)^n$.

We see that $z[n] = 2x[n-4] + 7y[n]$ . The **shift property** of the DFT is the following:

if $s[n] = x[n-n_0]$, then $S[k] = X[k]\,\mathrm{e}^{-\jmath(2\pi/N)kn_0}$.

Thus, using the shift property and the **linearity** of the DFT, we see that

$$Z[k] = 2X[k]\,\mathrm{e}^{-\jmath(2\pi/32)k4} + 7Y[k] = \begin{cases} 20 + \dfrac{7}{32}\dfrac{1-1/5^{32}}{1-(1/5)}, & k=0 \\ -10\mathrm{e}^{-\jmath(2\pi/32)4\cdot 12} + \dfrac{7}{32}\dfrac{1-1/5^{32}}{1-(1/5)\,\mathrm{e}^{-\jmath(2\pi/32)12}}, & k=12 \\ -10\mathrm{e}^{-\jmath(2\pi/32)4\cdot 20} + \dfrac{7}{32}\dfrac{1-1/5^{32}}{1-(1/5)\,\mathrm{e}^{-\jmath(2\pi/32)20}}, & k=20 \\ \dfrac{7}{32}\dfrac{1-1/5^{32}}{1-(1/5)\,\mathrm{e}^{-\jmath(2\pi/32)k}}, & \text{otherwise.} \end{cases}$$

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**So what?**

After we know how to compute the DFT of a signal, what can we do? There are an enormous number of applications; the DFT and its fast computational version, the FFT, are the foundation for much of signal processing.

- In lecture I demonstrated that we can perform audio signal compression by discarding frequency components with small DFT coefficients. This is the essence of how MP3 works.
- In lab you will see how to use the DFT to remove a contaminating tone from an audio signal.
- If we start with a continuous-time signal and sample it to form a discrete-time signal and then compute the DFT of that discrete-time signal, then we will soon discuss how the DFT coefficients are related to the spectrum of the original continuous-time signal. This is how instruments like digital oscilloscopes display the (approximate) spectrum of continuous-time signals.