
Ch. 8: IIR Filters

- Difference equation
- System function
- Frequency response
- Impulse response
- Poles/zeros
- Filter design

Reading

- Text Ch. 8

8.1

Introduction

In the preceding notch filter example we have seen that a system with poles that are not at the origin can be very useful.

Just as in the example that we worked out, such systems always have the property that the output signal value at any time n depends both on certain input signal values *as well as some previous output signal values*.

These systems are called **recursive**.

The general form for a (finitely-computable, causal, LTI) system that depends on both current and past inputs and past outputs is the following **difference equation**:

$$y[n] = \sum_{l=1}^N a_l y[n-l] + \sum_{k=0}^M b_k x[n-k].$$

Are systems having the above diffeq are **causal**, **linear**, and **time invariant**?

Yes. Causality is easy to see; LTI is not hard to show.

The b_k 's are called **feed-forward** coefficients.

The a_l 's are called **feedback** coefficients.

To implement such a system, computing each output signal value requires $N + M + 1$ multiplies. However, we usually say that the system is of N th order since the number of poles has great influence on the system properties.

When we design a filter, we get to pick N , M , and the **filter coefficients** a_1, \dots, a_N and b_0, \dots, b_M .

Usually we design in the z -plane though and work backwards from the zeros and poles to find these filter coefficients.

The above form is the *most general* type of system that we will consider in 206, and is the bread-and-butter of DSP work.

We consider only the usual case of real filters, so the filter coefficients, the a_l 's and b_k 's, are real numbers.

Are FIR filters a special case? Yes, just choose $N = 0$.

8.2

Impulse response: the hard way

The chapter is called "IIR" so apparently such systems (for $N \geq 1$) have an infinitely long impulse response, but that fact may not be immediately obvious.

Let's work out a concrete example now by "brute force." This is not the best way to find $h[n]$ for an IIR system in general!

Example. Consider the **first-order** system described by $y[n] = ay[n-1] + x[n]$, where a is a real number. Find the **impulse response** $h[n]$.

Before proceeding, we must make a very important assumption, called **initial rest conditions**.

- We assume the input is zero prior to some starting time n_0 , *i.e.*, $x[n] = 0$ for $n < n_0$. These are called **suddenly applied** inputs.
- We assume the output signal is zero prior to the starting time of the signal, *i.e.*, $y[n] = 0$ for $n < n_0$. We say that the system is **initially at rest**.¹

Note that $y[0] = ay[-1] + 2x[0]$ in this example. If the input signal $x[n]$ is zero for $n < 0$ (the usual case considered), then we assume $y[-1] = 0$ so $y[0] = 2x[0]$ for this example.

We assume **initial rest conditions** hereafter, both for simplicity of analysis and because that is the usual mode of operation of DSP systems. (All memory buffers are reset to zero when the system is powered up.)

¹The word "rest" has its origins in mathematical models for mechanical systems: think of a ball at rest on a flat plane that is subsequently subject to forces such as a swift kick. The term "at rest" is not so natural for digital systems, but we use it anyway for historical reasons.

By definition, the impulse response is the system output when the input is the unit impulse $x[n] = \delta[n]$.

In other words, the impulse response function satisfies the following recursive relationship: $h[n] = ah[n-1] + 2\delta[n]$. Because it is recursive, it is *not* a “final answer” for $h[n]$. However, we can “execute” the recursion to find the impulse response.

Using the initial rest conditions, we have that $h[n] = 0$ for $n < 0$ since the unit impulse input is zero for $n < 0$.

n	$\delta[n]$	$h[n-1]$	$h[n] = ah[n-1] + 2\delta[n]$
< 0	0	0	0
0	1	0	2
1	0	2	$2a$
2	0	$2a$	$2a^2$
3	0	$2a^2$	$2a^3$
\vdots	\vdots	\vdots	\vdots

By inspection we see that the **impulse response** is:

$$h[n] = \begin{cases} 2a^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases} = 2a^n u[n], \text{ where } u[n] \triangleq \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

We see that the impulse response decays to zero (if $|a| < 1$) but never reaches zero (for $a \neq 0$) so this is indeed an **IIR filter**.

Stability

What happens to $h[n]$ if $a > 1$ in the preceding example?

The impulse response “blows up” as $n \rightarrow \infty$.

A system is called **bounded-input bounded-output (BIB) stable** if all bounded input signals (inputs where $|x[n]| \leq c_1$ for some constant $c_1 < \infty$) produce output signals that are also bounded: $|y[n]| \leq c_2$ for some $c_2 < \infty$.

Is the unit impulse a bounded signal? Yes, with $c_1 = 1$.

In our example system, if $|a| > 1$ then the (bounded) unit impulse input produces an unbounded output a^n , so that system would not be BIBO stable.

What if $a = 1$? Would the system be stable in that case?

To show that the answer is “no” all we need is to find one bounded input signal that produces an unbounded output signal.

For this example, the step function input $x[n] = u[n]$ suffices.

n	$u[n]$	$y[n-1]$	$y[n] = y[n-1] + 2x[n]$
< 0	0	0	0
0	1	0	2
1	1	2	4
2	1	4	6
3	1	6	8
\vdots	\vdots	\vdots	\vdots

By inspection we see that the **step response** is:

$$y[n] = 2(n+1)u[n].$$

Is this bounded? No, so the system is unstable if $|a| = 1$ as well as if $|a| > 1$.

One can show that the system is stable if $|a| < 1$; we will return to that later when we discuss poles and zeros.

Numerically finding the impulse and step response

The above brute-force method only works for simple cases. For more complicated cases one can either use the **partial-fraction expansion method** described later, if an analytical solution is needed, or simply use MATLAB’s `filter` function if a numerical solution is adequate.

Using `filter` can be convenient for checking your analytical work.

8.3

System function

Before we can find more general tools for finding the impulse response of IIR filters, we must examine the system function.

Taking the z-transform of both sides of the general difference equation yields

$$Y(z) = \sum_{l=1}^N a_l z^{-l} Y(z) + \sum_{k=0}^M b_k z^{-k} X(z)$$

or equivalently

$$Y(z) \left[1 - \sum_{l=1}^N a_l z^{-l} \right] = \left[\sum_{k=0}^M b_k z^{-k} \right] X(z).$$

Thus the **system function** is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{l=1}^N a_l z^{-l}} = \frac{z^N}{z^M} \frac{\sum_{k=0}^M b_k z^{M-k}}{z^N - \sum_{l=1}^N a_l z^{N-l}} = \frac{z^N}{z^M} \frac{b_0 z^M + b_1 z^{M-1} + \dots + b_M z^0}{z^N - a_1 z^{N-1} - a_2 z^{N-2} - \dots - a_{N-1} z^1 - a_N}.$$

This is a ratio of polynomials so it is a **rational** system function.

Note that the *negatives* of the feed-forward coefficients appear in the denominator polynomial!

As before, the numerator polynomial will have M roots called **zeros** and the denominator polynomial will have N roots called **poles**. There can also be additional poles or zeros at the origin due to the z^N/z^M term. These roots do not affect the frequency response so are less important. In factored form the system function is:

$$H(z) = \frac{B(z)}{A(z)} = \frac{z^N \prod_{i=1}^M (z - z_i)}{z^M \prod_{j=1}^N (z - p_j)} = \frac{z^N (z - z_1)(z - z_2) \dots (z - z_M)}{z^M (z - p_1)(z - p_2) \dots (z - p_N)}.$$

Again we represent the system function graphically by its **pole-zero plot**.

Again, the relation between **frequency response** and **system function** is:

$$\boxed{\mathcal{H}(\hat{\omega}) = H(z) \Big|_{z=e^{j\hat{\omega}}} = H(e^{j\hat{\omega}})}$$

Example. For $y[n] = ay[n-1] + 2x[n]$ we have

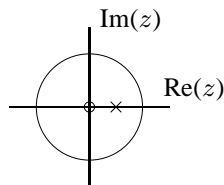
$$Y(z) = az^{-1}Y(z) + 2X(z)$$

or equivalently

$$Y(1 - az^{-1}) = 2X(z)$$

so the system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - az^{-1}} = 2 \frac{z}{z - a}.$$



So the pole-zero diagram is (for $a = 1/2$):

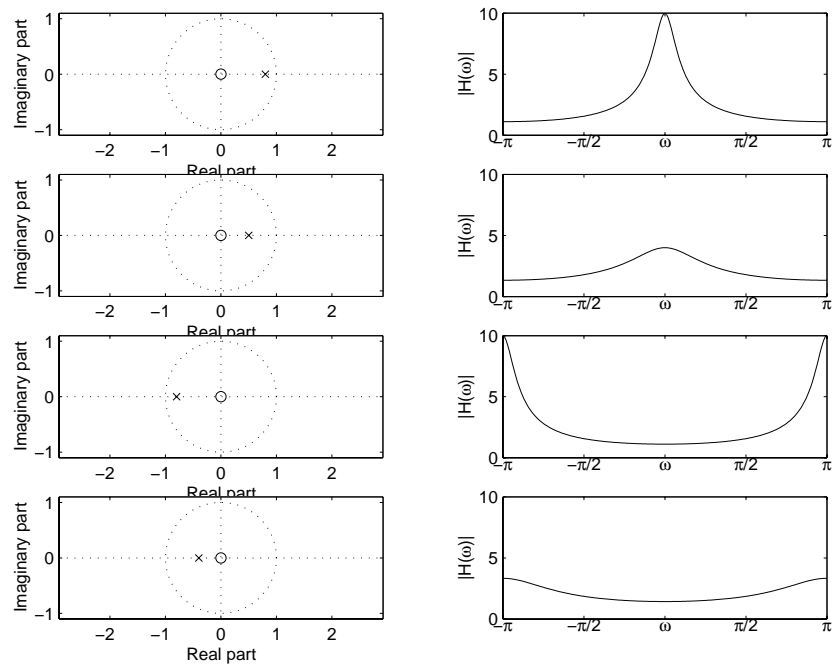
The frequency response is

$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = \frac{2}{1 - ae^{-j\hat{\omega}}},$$

so the magnitude response is

$$|\mathcal{H}(\hat{\omega})| = \frac{2}{\sqrt{(1 - ae^{-j\hat{\omega}})(1 - ae^{j\hat{\omega}})}} = \frac{2}{\sqrt{1 + a^2 - 2a \cos(\hat{\omega})}}.$$

It is difficult to visualize the frequency response from the formula for $\mathcal{H}(\hat{\omega})$, particularly for IIR filters. So we turn to `freqz`.



As expected from our previous analyses, frequencies that correspond to positions along the unit circle that are closer to the pole have a larger magnitude response.

Impulse response revisited

Given the system function $H(z) = \frac{2}{1-az^{-1}}$ for this example, can we work backwards to find the **impulse response** $h[n]$?

Recall the geometric series formula given in the lecture notes:

$$\sum_{n=0}^{\infty} b^n = \frac{1}{1-b}, \quad \text{if } |b| < 1.$$

Identify $b = az^{-1}$ in this example and we have

$$H(z) = \frac{2}{1-az^{-1}} = 2 \left[\sum_{n=0}^{\infty} (az^{-1})^n \right] = \sum_{n=0}^{\infty} 2a^n z^{-n}.$$

Since in general

$$H(z) = \sum_n h[n] z^{-n}$$

we see immediately that

$$h[n] = \begin{cases} 2a^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases} = 2a^n u[n].$$

This is valid only if $|b| = |az^{-1}| < 1$, *i.e.*, $|z| > |a|$. This is called the **region of convergence** of the z-transform, an important topic discussed in detail in EECS 451 but not in 206.

Of course we already knew this particular impulse response from the brute-force method used earlier, but using the z-transform will be more general.

However, the specific **series expansion** approach used above is convenient only in special cases. We still need to find a more general approach.

Nevertheless, we have established a particularly important z-transform pair that should be memorized:

$$a^n u[n] \iff \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

Discrete-time systems described by difference equations (FIR and IIR)

Difference equation:

$$y[n] = \sum_{l=1}^N a_l y[n-l] + \sum_{k=0}^M b_k x[n-k]$$

System function (in expanded polynomial and in factored polynomial forms):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 - \sum_{l=1}^N a_l z^{-l}} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 - a_1 z^{-1} - \dots - a_M z^{-N}} = z^{N-M} b_0 \frac{\prod_{k=1}^M (z - z_k)}{\prod_{k=1}^N (z - p_k)}$$

Relationships:

