Why study spectrum?

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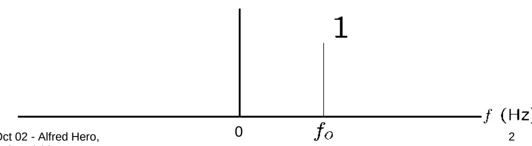
EECS206 F02

- Music synthesis, and other entertainment
- Long distance communications
- Matching a signal to channel, free-space, wire, CD, DVD, etc
- Recognition of speech/audio/video
- Filtering-out noise and disturbances
- Hiding a signal within another signal

Spectral Analysis of Continuous Time Signals

- Start with spectrum of a complex exponential:

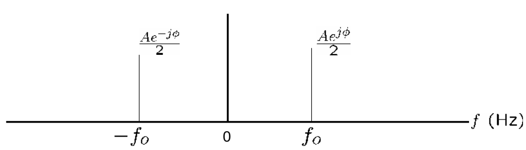
$$x(t) = e^{j\omega_o t} \qquad \omega_o = 2\pi f_o$$



Spectrum of Real Sinusoid

- Real sinusoidal signal

$$A\cos(\omega_o t+\phi) = A\frac{e^{j(\omega_o t+\phi)}+e^{-j(\omega_o t+\phi)}}{2} = \frac{Ae^{j\phi}}{2}e^{j\omega_o t}+\frac{Ae^{-j\phi}}{2}e^{-j\omega_o t}$$

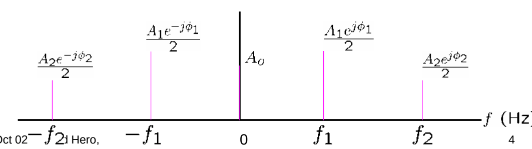


Note: Conjugate symmetry

Spectrum of many Sinusoids

- Superposition of sinusoids at different frequencies

$$A_o + A_1\cos(\omega_1 t+\phi_1) + A_2\cos(\omega_2 t+\phi_2)$$



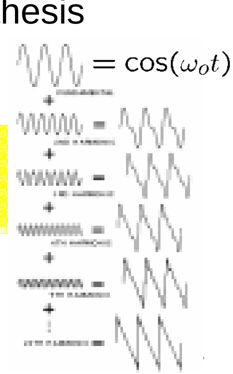
Fourier Synthesis

Combinations of sinusoids in just the right proportions yields a sawtooth wave ($\phi_o = 0$)

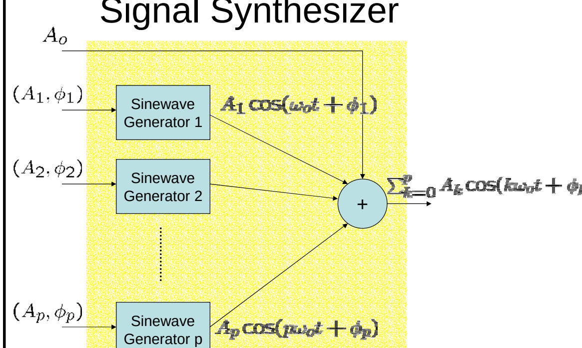
$$x_p(t) = \sum_{k=0}^{p} A_k\cos(\omega_k t+\phi_k)$$

k-th harmonic frequency:

$$\omega_k = k\omega_o$$



Signal Synthesizer

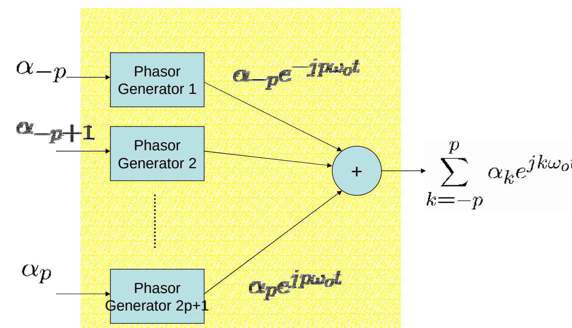


Phasor Form of Synthesizer

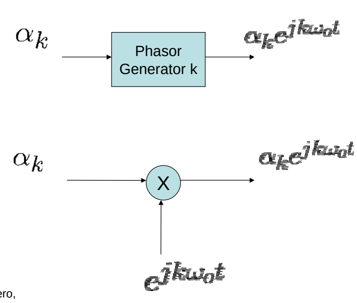
$$x_p(t) = \sum_{k=0}^{p} A_k \cos(k\omega_o t + \phi_k) = \sum_{k=0}^{p} A_k \left(\frac{e^{j(k\omega_0 t + \phi_k)} + e^{-j(k\omega_0 t + \phi_k)}}{2} \right) = \sum_{k=-p}^{p} \alpha_k e^{jk\omega_o t}$$

$$\alpha_k = \begin{cases} \frac{A_k e^{j\phi_k}}{2}, & k > 0 \\ A_o, & k = 0 \\ \frac{A_k e^{-j\phi_k}}{2}, & k < 0 \end{cases}$$

Signal Synthesizer



Phasor Generator: Multiplier Form



Fourier (Spectral) Analysis

- Is it possible to synthesize an arbitrary periodic signal by a sum of sinusoids?
- Yes! If x(t) is smooth and periodic with period:

$$T = 1/f_o = 2\pi/\omega_o$$

- Then x(t) can be represented as a sum of sinusoids at frequencies .

$$f_o, 2f_o, 3f_o, \ldots$$

Fourier Series

If x(t) is smooth and periodic with period $T = 2\pi/\omega_o$

$$x(t) = A_o + \sum_{k=1}^{\infty} A_k \cos(k\omega_o t + \phi_k)$$

$$x(t) = X_o + \Re e \left\{ \sum_{k=1}^{\infty} X_k e^{jk\omega_o t} \right\}$$

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_o t}$$

Fourier Series

Previous three FS formulas are equivalent

- Fourier coefficients

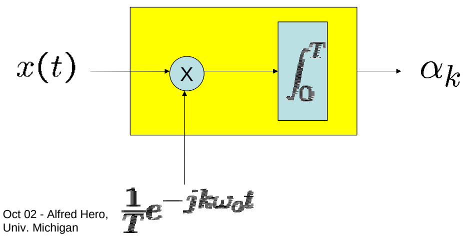
$$(A_k, \phi_k), X_k, \alpha_k$$

- Harmonic frequencies

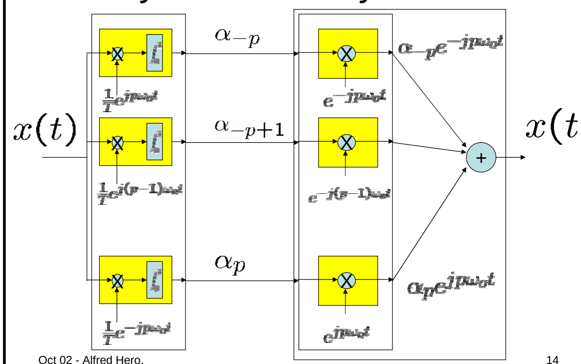
$$f_o, 2f_o, 3f_o, \ldots$$

Computing Fourier Coefficients

$$\alpha_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt$$



Analysis - Synthesis



What’s so special about it?

- This “filter bank: structure is ubiquitous in MP3, MPEG-4, JPEG-2000, and other digital standards.
- Now it remains to
  - Explain why Fourier series works
  - Do a couple of examples
  - Discuss some important properties
  - Extend to discrete time (sampled) signals