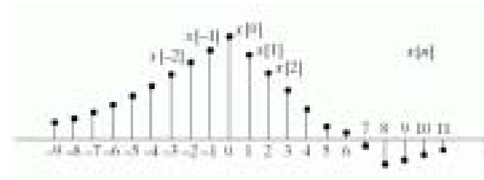
Spectrum Analysis for Discrete Time Signals

Prof Alfred Hero
EECS206 F02

- Just like continuous signals but with a few twists!
- Let's take things slowly…and start with a quick review of discrete time signals

Discrete Time Signals

$$x[n], \quad n = \ldots, -1, 0, 1, \ldots$$

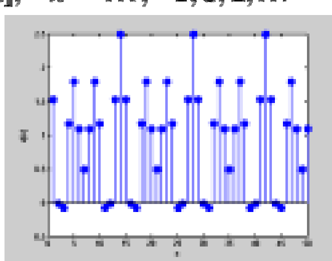


Periodic discrete-time signals

- x[n] periodic with period N satisfies

$$x[n+N] = x[n], \quad n = \ldots, -1, 0, 1, \ldots$$

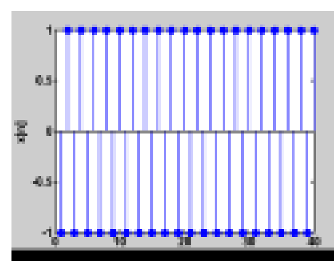
- Example 1:
- Period =N=13



- Example 2: x[n+2]=x[n]

$$x[n] = (-1)^n = \cos(\pi n)$$

- Period =N=2



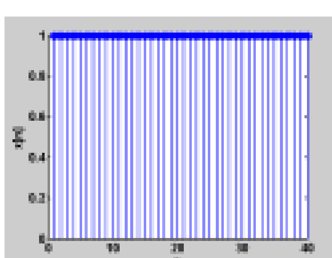
Example 3: Minimum possible value of N: N=1

$$x[n+1] = x[n]$$

- **This can only be a constant signal, e.g.** $x[n] = 1$

Period=N=1?

Convention:
$N = \infty$



Digital frequency

- For a periodic DT signal with fundamental period N its (digital) frequency is

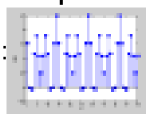
$$\hat{\omega} = 2\pi/N, \quad \text{(rads/sample)}$$

- or

$$\hat{f} = 1/N, \quad \text{(cycles/sample)}$$

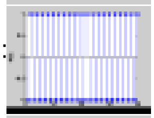
Examples

- Periodic signal 1:



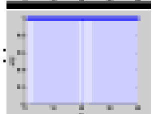
$\hat{\omega} = 2\pi/13$

- Periodic signal 2:



$\hat{\omega} = 2\pi/2$

- Periodic signal 3:



$\hat{\omega} = 0, 2\pi$

Interesting observation

- It appears that the digital frequency for a DT signal has to satisfy

$$0 \leq \hat{\omega} \leq 2\pi$$

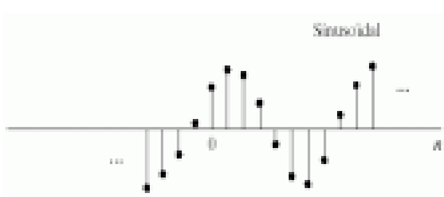
- Or

$$0 \leq \hat{f} \leq 1$$

- Q. Is this true?
- A. No…need to study DT sinusoids to see why

Discrete time sinusoids

$$x[n] = A\cos(\omega_o n + \phi), \quad n = \ldots, -1, 0, 1, \ldots$$



Not all DT sinusoids are periodic!

- Consider DT sinusoid $x[n] = A\cos(\omega_o n)$
- For period N require $x[n+N] = x[n]$ or

$$\cos(\omega_o(n+N)) = \cos(\omega_o n + \omega_o N)$$

- Thus there must exist integer M such that:

$$\omega_o N = 2\pi M \quad \text{or} \quad \omega_o = 2\pi \underbrace{\frac{M}{N}}_{f_o}$$

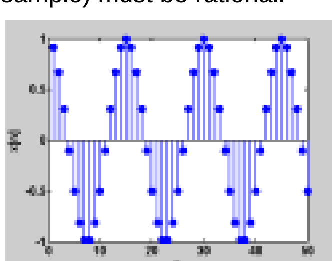
- Condition for DT sinusoid to be periodic:

$$f = \hat{f} = M/N$$

- i.e. freq (cycles/sample) must be rational.
- Example

$x[n] = \cos(2\pi/15n)$

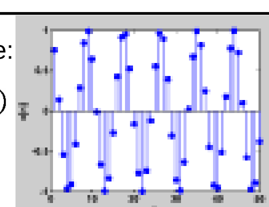
$\hat{f} = 1/15$



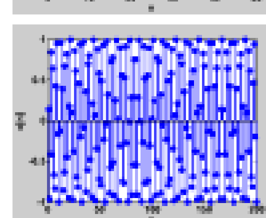
- Non-Periodic example:

$x[n] = \cos(2\pi\sqrt{\pi}/2n)$

- After 50 samples…



…after 200 samples



Periodic DT Sinusoids are Doubly Periodic

- If the period is N then for any integer m

$$\cos(\hat{\omega}_o(n+mN)) = \cos(\hat{\omega}_o n)$$

- and

$$\begin{aligned}\cos((\hat{\omega}_o + m2\pi)n) &= \cos(\hat{\omega}_o n + mn2\pi)\\ &= \cos(\hat{\omega}_o n)\end{aligned}$$

- Conclude: it's periodic in frequency too!

$\hat{\omega}_o$ and $\hat{\omega}_o + m2\pi$ are equivalent

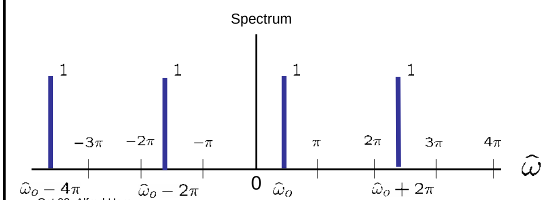
$\hat{f}_o$ and $\hat{f}_o + m$ are equivalent

DT Spectrum Analysis

- First consider a DT complex exponential

$$x[n] = e^{j\hat{\omega}_o n}$$

$$\hat{\omega}_o = 2\pi M/N \le 2\pi \text{ rads/sample}$$

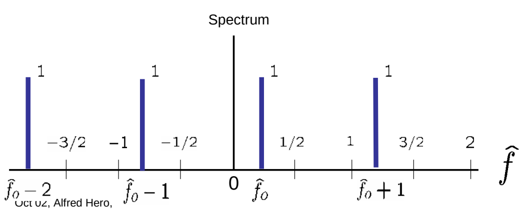


DT Spectrum Analysis(ctd)

- Equivalently, in cycles/sample

$$x[n] = e^{j2\pi\hat{f}_o n}$$

$$\hat{f}_o = M/N \le 1 \text{ cycles/sample}$$

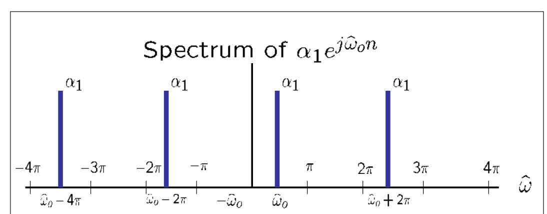


Spectrum of DT periodic sinusoid

- Use inverse Euler to obtain

$$A\cos(\hat{\omega}_o n + \phi) = \alpha_1 e^{j\hat{\omega}_o n} + \alpha_{-1} e^{-j\hat{\omega}_o n}$$

$$\alpha_1 = \alpha_{-1}^* = \frac{Ae^{j\phi}}{2}$$



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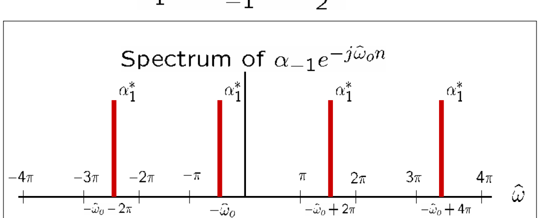

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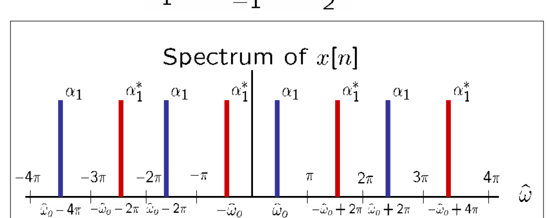

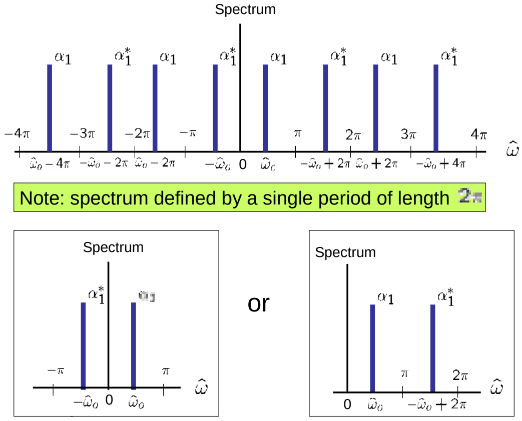

DT Fourier Series for Periodic Signals

- For any periodic DT signal with period N:

$$x[n] = \sum_{k=0}^{N-1} \alpha_k e^{jk\hat{\omega}_o n}$$

$$\alpha_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\hat{\omega}_o n}$$

$$\hat{\omega}_o = 2\pi/N$$

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- Alternative form for DT Fourier series:

Discrete Fourier Transform (DFT) pair:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k}{N}n}$$ Inverse DFT

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$ DFT

$X[k] = \alpha_k$, $n = 0, \ldots, N-1$, are DFT coefficients

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Some properties of DT FS

- DT FS is a finite number of sinusoids
- Parseval relation:

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |X[k]|^2$$

- For real DT signal: Conjugate symmetry

$$X[k] = X^*[N-k]$$

This follows from csymmetry of spectrum about $\hat{\omega} = \pi$ (or $\hat{f} = 1/2$)

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Important Facts about DFT

- DFT represents N successive samples of *any* signal, periodic or not.
- DFT requires only finite number N of complex exponentials to represent N samples of signal
- There is a very fast DFT algorithm called FFT when $N = 2^M$
- The DFT coefficients inherit many properties of FS coefficients (Parseval, etc.)
- For real periodic DT signals x[n] and N = even:

Only need specify $X[0], \ldots, X[N/2]$ since $X[k] = X^*[N-k]$

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