EECS 206F01: EXAM #3 - 12/17/01

Solutions

What follows are restatements of each exam question, the correct answers, strategies for working each of the problems, and partial credit guidelines. Please look over this material carefully to make sure that you received the credit you deserve for the work you have shown.

Regrading procedure

After you've read through the partial credit guidelines for a given problem, if you believe you should have received more points than given, please write up your petition along with your reasoning and hand the petition *along with your original exam* to either Prof. Neuhoff or Wakefield.

Please remember that the amount of partial credit given for each problem differs according to the problem and the nature of the mistakes. *You may believe that more partial credit should have been given for the work you have shown*, but if we indicate no more than 5 points for a particular body of work, you will receive no more than 5 points, regardless of your petition.

Also, please remember that some answers received NO partial credit, no matter how much work was performed. This reflects the fact that some of the incorrect answers were wrong on fundamental grounds, rather than particular aspects of the problem. A petition requesting that work shown in support of a fundamental error will be rejected automatically.

You have until the 1/18/02 to submit your petitions for regrade in writing.

Please note: in the case of Exam 3, an initial triage will be performed on your petition. If your regrade would not change your letter grade, the petition will not be evaluated further. If your regrade would change your letter grade, it will be considered further.

1. Let $x[n] = \cos\left(2\pi\frac{1}{5}n\right)$, $y[n] = x[2n]$, and $w[n] = x[2n+1]$. Then

a. y[n] and w[n] have the same frequency, which is twice that of x[n]. True

x[n] has frequency $\frac{1}{5}$

$y[n] = x[2n] = \cos\left(2\pi\frac{1}{5}2n\right)$ which has frequency $\frac{2}{5}$

$w[n] = x[2n+1] = \cos\left(2\pi\frac{1}{5}(2n+1)\right) = \cos\left(2\pi\frac{2}{5}n+2\pi\frac{1}{5}\right)$ which also has frequency $\frac{2}{5}$

b. y[n] and w[n] have different frequencies. **False**

c. y[n] and w[n] have the same frequency and the same phase. **False**

d. y[n] and w[n] have different amplitudes **False**

e. None of the above **False**

Partial Credit:

+5 for finding correct formulas for y[n] and w[n]

up to 7 points if the approach is correct and wrong answer is due to a numerical error.

2. Given a real-valued periodic signal x[n] with period 42, let X[k] be the 50 pt DFTof x[1],x[2],...,x[50]. If

$$X[8] = 0.5\, e^{j0.2}$$

then

a. the amplitude of X[42] = -0.5 **False**.

Amplitude is never negative. Moreover, recall that $X[N-k] = X^*[k]$ for any k.

Here, N = 50. And This implies $X[42] = X[50-8] = X^*[8]$.

Therefore, $|X[42]| = |X^*[8]| = |X[8]| = 0.5$.

b. the angle of X[42] is -0.2 True

Since $X[42] = X[50-8] = X^*[8]$, angle of X[42] = - angle of X[8] = -0.2

c. the angle of X[34] is -0.2 **False**

We have no information about X[34]

d. a 50-pt DFT can't exist because the period of the signal is 42 **False**

As discussed in class and lab, it is perfectly acceptable and often useful to apply the N-point DFT to an arbitrary set of N numbers.

e. None of the above is true **False**

Partial Credit:

up to +5 if you answered c. and gave a plausible explanation for why

+4 pts if you wrote $X[N-k] = X^*[k]$

+2 if you didn't get any of the above partial credit but wrote the DFT synthesis or analysis formulas or both.

3. The signal $x[n] = \begin{cases} 2, & n = 0 \text{ or } n = \text{multiple of } 3 \\ 0, & n \neq \text{multiple of } 3 \end{cases}$ is the input to a filter described by the difference equation

$$y[n] = \frac{1}{2} y[n-1] + x[n]$$

The 6-point DFT of the output is

a. $Y[k] = \frac{1}{3} \frac{1+(-1)^k}{1 - \frac{1}{2} e^{-j\frac{\pi}{3}k}}$, $k = 0,...,5$ **True**

b. $Y[k] = 2 \frac{1}{1 - \frac{1}{2} e^{-j\frac{\pi}{3}k}}$, $k = 0,...,5$ **False**

c. $Y[k] = 2 \frac{1}{1 - \frac{1}{2} e^{-j\frac{2\pi}{3}k}}$, $k = 0,...,5$ **False**

d. $Y[k] = \frac{2}{3} \frac{1}{1 - \frac{1}{2} e^{-j\frac{2\pi}{3}k}}$, $k = 0,...,5$ **False**

e. $Y[k] = \frac{1}{3} \frac{1}{1 - \frac{1}{2} e^{-j\frac{2\pi}{3}k}}$, $k = 0,...,5$ **False**

Solution: This problem is similar to Problem 8 of Exam 2. Note that since $x[n]$ is periodic we need to use the DFT approach rather than the z=transform approach. The key fact is

$$Y[k] = X[k]\, H(e^{j\frac{2\pi}{6}k})$$

where $X[k]$ is the 6-point DFT of $x[n]$, $H(z)$ is the system function, and $H(e^{j\hat{\omega}})$ is the frequency response of the system.

$$X[k] = \frac{1}{6} \sum_{n=0}^{5} x[n]\, e^{-j\frac{2\pi}{6}kn} = \frac{1}{6} x[0]\, e^{-j\frac{2\pi}{6}k0} + \frac{1}{6} x[3]\, e^{-j\frac{2\pi}{6}k3}$$ since only x[0] and x[3] are not zero

$$= \frac{1}{6} 2 + \frac{1}{6} 2\, e^{-j\pi k} = \frac{1}{3} (1+(-1)^k)$$

From the difference equation we see, $Y(z) = \frac{1}{2} z^{-1} Y(z) + X(z)$.

Therefore, $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}$ and $H(e^{j\frac{2\pi}{6}k}) = \frac{1}{1 - \frac{1}{2} e^{-j\frac{2\pi}{3}k}}$

Substituting the expressions for $X[k]$ and $H(e^{j\frac{2\pi}{6}k})$ yields a.

Partial Credit:

+3 for writing $Y[k] = X[k]\, H(e^{j\frac{2\pi}{6}k})$ where X and H are identified

+3 for finding $H(e^{j\frac{2\pi}{6}k}) = \frac{1}{1 - \frac{1}{2} e^{-j\frac{\pi}{3}k}}$ or +2 for $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}$

+3 for finding $X[k] = \frac{1}{3} (1+(-1)^k)$ or +1 for defintion of $X[k]$

+7 if you answered d and worked out the answer correctly except that you used $N = 3$ instead of $N = 6$

4. The signal

$$x[n] = 5\cos(\frac{\pi}{4}n + 1) + 4\cos(\frac{\pi}{6}n) + 3$$

is the input to an FIR filter with coefficients

$$\{b_k\} = \{1, -\sqrt{2}, 1\} .$$

The output of the filter is

a. $y[n] = 3.62\cos(\frac{\pi}{4}n + 1.31) + 1.27\cos(\frac{\pi}{6}n - .52) + 1.76$ **False**

b. $y[n] = 3.62\cos(\frac{\pi}{4}n + 1.31) + 1.76$ **False**

c. $y[n] = 1.27\cos(\frac{\pi}{6}n - .52) + 1.76$ True

d. $y[n] = x[n]\,H(\hat{\omega})$ **False**

e. More than one of the above **False**

Solution: This problem is like Problem 7 of Exam 2.

The approach to this problem is to find the frequency response $H(\hat{\omega})$ of the FIR filter and then to use linearity of the FIR filter and the fact that if the input is $A\cos(\hat{\omega}n+\phi)$, then the output is $A|H(\hat{\omega})|\cos(\hat{\omega}n+\phi+\text{angle}(H(\hat{\omega})))$. Specifically, for the given input $x[n]$, the output is

$$y[n] = 5\,|H(\tfrac{\pi}{4})|\cos(\tfrac{\pi}{4}n + 1 + \text{angle}(H(\tfrac{\pi}{4}))) + 4\,|H(\tfrac{\pi}{6})|\cos(\tfrac{\pi}{6}n + \text{angle}(H(\tfrac{\pi}{6}))) + 3\,|H(0)|$$

The frequency response is $H(\hat{\omega}) = 1 - \sqrt{2}\,e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$. Thus,

$$H(\tfrac{\pi}{4}) = 1 - \sqrt{2}\,e^{-j\frac{\pi}{4}} + e^{-j\frac{\pi}{2}} = 0, \quad H(\tfrac{\pi}{6}) = 1 - \sqrt{2}\,e^{-j\frac{\pi}{6}} + e^{-j\frac{\pi}{3}} = .318\,e^{-j0.52}, \quad H(0) = 1 - \sqrt{2} + 1 = .586$$

Therefore,

$$y[n] = 5\times 0\cos(\tfrac{\pi}{4}n + 1) + 4\times .318\cos(\tfrac{\pi}{6}n - 0.52) + 3\times 0.586$$

$$= \mathbf{1.27\cos(\tfrac{\pi}{6}n - 0.52) + 1.76}$$

Notice that once you discover that $H(\frac{\pi}{4}) = 0$, you could immediately see that the answer does not have a component at frequency $\frac{\pi}{4}$, which eliminates answers a. and b.

Very Important Note: As mentioned in the solution to Exam 2, $y[n] = x[n]\,H(\hat{\omega})$ **holds when and only when x[n] is a complex exponential.** Since $x[n]$ in this problem is not a complex exponential, d. is incorrect.

Partial Credit:

+ 6 if you answer e and correctly show c is true and also claim d is true.

+ 3 for writing input = $A\cos(\hat{\omega}n+\phi) \Rightarrow$ output = $A|H(\hat{\omega})|\cos(\hat{\omega}+\phi+\text{angle}(H(\hat{\omega})))$

+5 for computing $H(\frac{\pi}{4})$, $H(0)$, $|H(\frac{\pi}{6})|$ and $\text{angle}(H(\frac{\pi}{6}))$

5. A signal x[n] is the input to a filter with frequency response

$$H(\hat{\omega}) = 2 - e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}.$$

Such filtering has the effect of performing sliding-correlation with what signal? (Neither the filtering nor the correlation involves mean subtraction.)

a. $p[n] = 2\delta[n] - \delta[n-2] + 2\delta[n-3]$

b. $p[n] = \delta[n] - \delta[n-2] + 2\delta[n-3]$

c. $p[n] = \delta[n] - \delta[n-1] + 2\delta[n-4]$ correct answer

d. $p[n] = 2\delta[n] - \delta[n-3] + \delta[n-4]$

e. none of the above

Solution: To do sliding correlation of signal x[n] with pulse

$$p[n] = p[0]\,\delta[n] + p[1]\,\delta[n-1] + \ldots + p[M]\,\delta[n-M]$$

we use an IIR filter with impulse response

$$h[n] = p[M-n] = p[M]\,\delta[n] + p[M-1]\,\delta[n-1] + \ldots + p[0]\,\delta[n-M]$$

Conversely, a filter with impulse response

$$h[n] = h[0]\,\delta[n] + h[1]\,\delta[n-1] + \ldots + h[M]\,\delta[n-M]$$

performs sliding correlation with the pulse

$$p[n] = h[M-m] = h[M]\,\delta[n] + h[M-1]\,\delta[n-2] + \ldots + h[0]\,\delta[n-M]$$

In this problem, from $H(\hat{\omega})$ we see that

$$h[n] = 2\delta[n] - \delta[n-3] + \delta[n-4] .$$

From this we see that M = 4, and

$$p[n] = \delta[n] - \delta[n-1] + 2\delta[n-4]$$

Partial Credit:

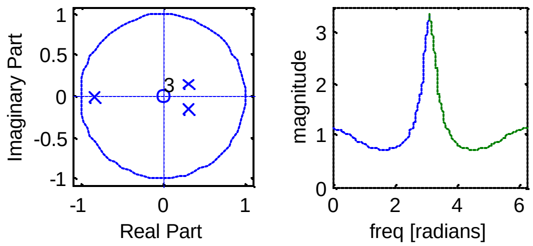
+2 for finding h[n]

+3 for h[n] = p[M-n]

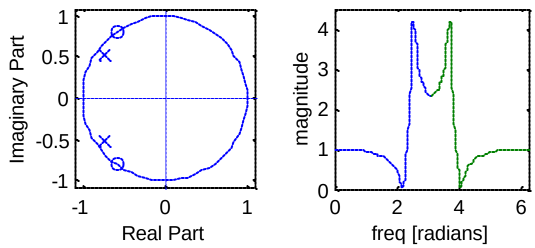
+2 for finding M = 4

6. Which of the following pole/zero plots and magnitude spectra are **inconsistent**?

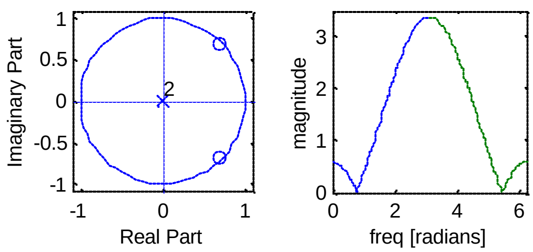
a.



b.



c.



d. More than one of the above

e. None of the above

Answer: e

Solution:

Option a) has a pole near the frequency π, where there is an increase in the spectrum. The conjugate pole pair lies closer to the origin and a frequency of 0. Therefore, we should expect a small increase in the magnitude spectrum that is observed near 0.

Option b) has a conjugate pole pair inside the unit circle centered at frequencies just a little larger than 0.75π and a conjugate zero pair close to, if not exactly on, the unit circle centered at frequencies just a little smaller than 0.75π. This is consistent with the null and peak observed between 2 and 3 radians.

Option c) has a conjugate zero pair close to, if not exactly on, the unit circle centered at frequencies about 0.25π. This is consistent with the null in the spectrum observed in the neighborhood of 0.25π.

Partial credit:

Up to 5 points were given if you properly interpreted the pole/zero plots but incorrectly interpreted the range of frequency as being from 0 to 2π.

A few of you misread the word inconsistent, argued that each of the pairs was consistent, and chose c). Up to 9 points were awarded for this mistake in reading.

Generally, between 3 and 5 points were awarded for interpretations that missed some of the key features of the pairs in explaining apparent discrepancies.

7. If $X(z) = z^{-4}(1 + 2z + 0.25z^2)$ and $Y(z) = 10z^2 X(z)$, then

a. $y[n] = z^2(2.5\delta[n-2] + 20\delta[n-3] + 10\delta[n-4])$

b. $y[n] = 2.5\delta[n] + 20\delta[n-1] + 10\delta[n-2]$

c. $y[n] = 10z^{-2}(\delta[n] + 2\delta[n+1] + 0.25\delta[n+2])$

d. $y[n] = 10\delta[n] + 20\delta[n+1] + 2.5\delta[n+2]$

e. None of the above

Answer: b

Solution: Substituting the specific form of $X(z)$ and expanding, we have that

$$\begin{aligned} Y(z) &= 10z^2 X(z) \\ &= 10z^{-2}(1 + 2z + 0.25z^2) \\ &= 10z^{-2} + 20z^{-1} + 2.5z^0 \end{aligned}$$

Therefore, $y(n)$ is the finite-length sequence of delta functions, e.g.,

$$y(n) = 2.5\delta(n) + 20\delta(n-1) + 10\delta(n-2)$$

Partial credit:

Options a) and c) are wrong, foundationally! Both represent an impossible mixing of sequences with Z-transforms, which is a bit like mixing water with oil. Sequences eliminate the Z terms, Z-transforms eliminate the n terms, and there is no in-between. Efforts showing either of these options were given up to 2 points for set up, and that's it.

Option d) was given up to 5 points credit, depending on the amount of work that was shown. Basically, this answer results if you have messed up on ordering the coefficients and/or handling the negative powers of z properly.

Option e) was given up to 5 points credit for work that attempted to show b) to be wrong, otherwise 2 points were given.

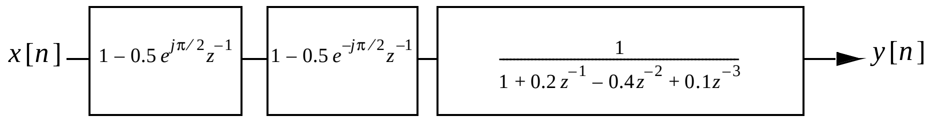
8. Given the LTI system H

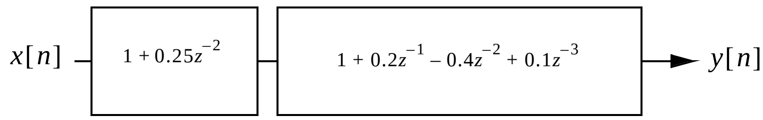
$$y[n] = -0.2y[n-1] + 0.4y[n-2] - 0.1y[n-3] + x[n] + 0.25x[n-2]$$

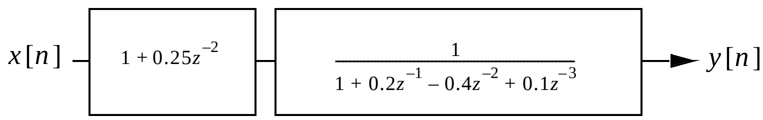
with two of its zeros at

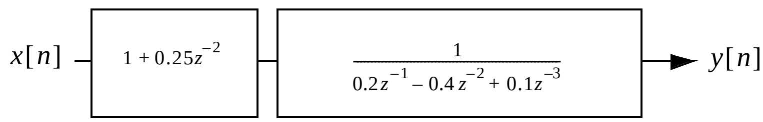
$$0.5e^{j\pi/2},\ 0.5e^{-j\pi/2}$$

which of the following implements H as a cascade of LTI systems with real coefficients?

a. 

b. 

c. 

d. 

e. None of the above

Answer: c

Solution: Z-transform both sides of the difference equation

$$y[n] = -0.2y[n-1] + 0.4y[n-2] - 0.1y[n-3] + x[n] + 0.25x[n-2]$$

$$\therefore \quad Y(z) = -0.2z^{-1}Y(z) + 0.4z^{-2}Y(z) - 0.1z^{-3}Y(z) + X(z) + 0.25z^{-2}X(z)$$

and form the system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.25z^{-2}}{1 + 0.2z^{-1} - 0.4z^{-2} + 0.1z^{-3}}$$

We recognize that the system function can be factored as

$$H(z) = (1 + 0.25z^{-2})\frac{1}{1 + 0.2z^{-1} - 0.4z^{-2} + 0.1z^{-3}}$$

and that the coefficients in each sub-system are real. Therefore, using the fact that the system functions of cascaded systems multiply, c) is the correct answer.

Partial Credit:

Option a). Up to 7 points were given for showing a) to be the answer but failing to recognize the requirement that the coefficients of each system block be real.

Options b), d), and e). Up to 5 points were given for setting up the problem and beginning to develop a solution. In particular, if you reached the proper form for $H(z) = \frac{B(z)}{A(z)}$, you received up to five points. If you did not reach this point, you were given up to three points.

9. Given the impulse response of a LTI system H

$$h[n] = 5(0.25^n)\cos(0.2\pi n)u[n] + 10(0.9^n)u[n]$$

where $u[n]$ is the unit-step function.

a. $H(z)$ has 3 poles and no zeros

b. $H(z) = \dfrac{5}{(1-0.25e^{-j0.2\pi}z^{-1})(1-0.25e^{j0.2\pi}z^{-1})} + \dfrac{10}{(1-0.9z^{-1})}$

c. $H(z) = \dfrac{5(1-0.9z^{-1})+10}{(1-0.25e^{-j0.2\pi}z^{-1})(1-0.25e^{j0.2\pi}z^{-1})(1-0.9z^{-1})}$

d. More than one of the above is correct

e. None of the above is correct

Answer: e

Solution: The key to solving the problem is to recognize that the specific form of the impulse response is a sum of impulse response, as found in the course material on the inverse Z transform:

$$h[n] = \sum_{k=1}^{K} A_k p_k^n u[n]$$

where the p_k are the poles of the system and the A_k are the coefficients obtained through partial-fraction expansion. That is,

$$H(z) = \sum_{k=1}^{K} \frac{A_k}{1-p_k z^{-1}}$$

In the text and in class, we developed special cases for real and complex-conjugate pair poles. If you recalled those, you could skip a few of the following steps. If not, we proceed by expanding the cosine

$$\cos(0.2\pi n) = \frac{e^{j0.2\pi n} + e^{-j0.2\pi n}}{2}$$

so that

$$h[n] = \frac{5}{2}(0.25e^{j0.2\pi})^n u[n] + \frac{5}{2}(0.25e^{-j0.2\pi})^n u[n] + 10(0.9^n)u[n]$$

Converting and simplifying

$$H(z) = \frac{2.5}{1-0.25e^{j0.2\pi}z^{-1}} + \frac{2.5}{1-0.25e^{-j0.2\pi}z^{-1}} + \frac{10}{1-0.9z^{-1}}$$

$$= \frac{5(1-Re\{0.25e^{j0.2\pi}\}z^{-1})}{(1-0.25e^{j0.2\pi}z^{-1})(1-0.25e^{-j0.2\pi}z^{-1})} + \frac{10}{1-0.9z^{-1}}$$

At this point, we need go no further. Option b) is wrong, since the numerator of the first term in the summand is wrong. Option c) is also wrong. The denominator is correct, but when cross-multiplying the equation above, a different expression obtains for the numerator. Finally, Option a) is also wrong: the numerator is a non-trivial polynomial and, therefore, has zeros. Therefore, none of the above options is correct and the proper answer to the problem is e).

Partial credit:

Option a) could earn no more than 1 point for the set-up alone. This is a foundational error. You should make sure you understand the correct answer and how zero's are truly present.

Options b) and c) were credited up to 5 points for development beyond the initial set-up and first steps of solution. Within this crediting, up to two points were given for recognizing the poles, and up to three points were given for working the problem using the proper Z-transform, partial-fraction forms.

10. Given that

$$y[n] = -\sum_{k=1}^{K} a_k y[n-k] + \sum_{l=0}^{L} b_l x[n-l]$$
$$w[n] = -\sum_{k=1}^{K} c_k w[n-k] + \sum_{l=0}^{L} d_l y[n-l]$$

then $\sum_k \alpha_k w[n-k] = \sum_l \beta_l x[n-l]$ where, for all k and l,

a. $\alpha_k = a_k * c_k$ and $\beta_l = b_l * d_l$, where the convolved sequences include $a_0 = 1$, $c_0 = 1$

b. $\alpha_k = a_k * c_k$ and $\beta_l = b_l * d_l$, where a_0 and c_0 do not exist in the convolved sequences

c. $\alpha_k = c_k$ and $\beta_l = b_l$

d. $\alpha_k = a_k + c_k$ and $\beta_l = b_l + d_l$

e. $\alpha_k = a_k c_k$ and $\beta_l = b_l d_l$, where the multiplied sequences include $a_0 = 1$, $c_0 = 1$

Answer: a

Solution: The easiest way to work this problem is to pull the terms of the first summation of each right hand side to the left hand side:

$$y[n] + \sum_{k=1}^{K} a_k y[n-k] = \sum_{l=0}^{L} b_l x[n-l]$$
$$w[n] + \sum_{k=1}^{K} c_k w[n-k] = \sum_{l=0}^{L} d_l y[n-l]$$

and combining the isolated $y[n]$ and $w[n]$ terms

$$\sum_{k=0}^{K} a_k y[n-k] = \sum_{l=0}^{L} b_l x[n-l]$$

$$\sum_{k=0}^{K} c_k w[n-k] = \sum_{l=0}^{L} d_l y[n-l]$$

where $a_0 = 1, c_0 = 1$. Taking Z transforms of all four summations, we have that

$$\sum_{k=0}^{K} a_k z^{n-k} Y(z) = \sum_{l=0}^{L} b_l z^{n-l} X(z)$$

$$\sum_{k=0}^{K} c_k z^{n-k} W(z) = \sum_{l=0}^{L} d_l z^{n-l} Y(z)$$

Pulling the Z transforms of the input and output sequences to the left hand side of each equation, we obtain

$$\frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{L} b_l z^{n-l}}{\sum_{k=0}^{K} a_k z^{n-k}} = \frac{B(z)}{A(z)}$$

$$\frac{W(z)}{Y(z)} = \frac{\sum_{l=0}^{L} d_l z^{n-l}}{\sum_{k=0}^{K} c_k z^{n-k}} = \frac{D(z)}{C(z)}$$

where $A(z), B(z), C(z)$ and $D(z)$ denote the polynomials with coefficients $\{a_k\}, \{b_k\}, \{c_k\}$ and $\{d_k\}$, respectively. From these equations, we find a relationship between $W(z)$ and $X(z)$ as follows:

$$W(z) = \frac{D(z)}{C(z)} Y(z)$$

$$Y(z) = \frac{B(z)}{A(z)} X(z)$$

$$\therefore \quad W(z)A(z)C(z) = X(z)B(z)D(z)$$

Recognizing that multiplication in the Z domain is the same as convolution in the sequence domain, e.g.,

$$A(z)C(z) \leftrightarrow a_k * c_k$$

$$B(z)D(z) \leftrightarrow b_k * d_k$$

we conclude that the sequence α_k is the convolution $a_k * c_k$ and the sequence β_k is the convolution $b_k * d_k$. Noting the condition, $a_0 = 1, c_0 = 1$, used in developing the solution, we see that option a) is correct.

Partial credit:

Up to two points were given for the initial set-up of the problem in a manner that would lead to some reasonable form of solution. Beginning to work the solution earned up to one more point. Successfully working through the Z-transforms to achieve the form $B(z)D(z)/A(z)C(z)$ earned up to a total of 5 points. Finally, successfully solving the problem by showing all the work but failing to handle the $a_0 = 1, c_0 = 1$ condition earned up to 8 points.