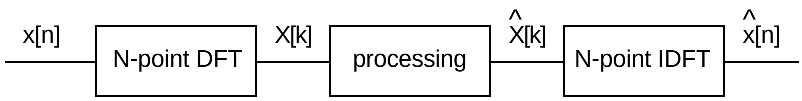
Notes: The Distortion of Transform Coding

We here derive the fact, used in Lab 5, that the overall MSE of a transform coder equals the sum of the MSE's of the scalar quantizers. As we will see, this is basically a corollary to Parseval's relation.

Parseval's Relation: (Property 13, Sec. 4.2.3, Lab 4) If $X[0],...,X[N-1]$ is the N-point DFT of $x[0],...,x[N-1]$, then

$$\sum_{k=0}^{N-1} |X[k]|^2 = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n] \ .$$

Transform Processing: Let us begin by considering a generic DFT-based "transform processing system" shown below, of which the transform coder considered in Lab 5 can be viewed as a special case.



Fact 1: The MSE between the input $x[0],...,x[N-1]$ and the output $\hat{x}[0], ..., \hat{x}[N-1]$ of a transform processing system equals the sum of the MSE's between the DFT coefficients $X[0],...,X[N-1]$ and the output $\hat{X}[0], ..., \hat{X}[N-1]$. That is,

$$\text{MSE} \ = \ \frac{1}{N} \sum_{n=0}^{N-1} \left(x[n]-\hat{x}[n]\right)^2 = \sum_{k=0}^{N-1} |X[k]-\hat{X}[k]|^2 \ .$$

Derivation: We first observe that in the above diagram $\hat{X}[0], ..., \hat{X}[N-1]$ is the DFT of $\hat{x}[0], ..., \hat{x}[N-1]$. Moreover, since $X[0],...,X[N-1]$ is the DFT of $x[0],...,x[N-1]$, we conclude from the linearity of the DFT (Property 6, Sec. 4.2.3, Lab 4) that

$$\left(X[0],...,X[N-1] - \hat{X}[0], ..., \hat{X}[N-1]\right) \quad \text{is the DFT of} \quad \left(x[0],...,x[N-1] - \hat{x}[0], ..., \hat{x}[N-1]\right)$$

Therefore, Parsevals Relation shows

$$\sum_{k=0}^{N-1} |X[k]-\hat{X}[k]|^2 = \frac{1}{N} \sum_{n=0}^{N-1} \left(x[n]-\hat{x}[n]\right)^2 .$$

The coefficients in Lab 5: In Lab 5, we choose $N = 8$, and we convert $X[0],...,X[7]$ to $c[0],...,c[7]$, where

$$c[0] = X[0], \quad c[1] = \sqrt{2}\ \text{Re}\{X[1]\}, \quad c[2] = \sqrt{2}\ \text{Re}\{X[2]\}, \quad c[3] = \sqrt{2}\ \text{Re}\{X[3]\},$$

$$c[4] = X[4], \quad c[5] = \sqrt{2}\ \text{Im}\{X[1]\}, \quad c[6] = \sqrt{2}\ \text{Im}\{X[2]\}, \quad c[7] = \sqrt{2}\ \text{Im}\{X[3]\}.$$

The $c[k]$'s are a kind of modified DFT. Notice that the $X[k]$'s can be recovered from the $c[k]$'s via

$$X[0] = c[0], \quad X[1] = \frac{1}{\sqrt{2}}(c[1]+jc[5]), \quad X[2] = \frac{1}{\sqrt{2}}(c[2]+jc[6]), \quad X[3] = \frac{1}{\sqrt{2}}(c[3]+jc[7]),$$

$$X[4] = c[4], \quad X[5] = X^*[3] = \frac{1}{\sqrt{2}}(c[3]-jc[7]), \quad X[6] = X^*[2] = \frac{1}{\sqrt{2}}(c[2]-jc[6]),$$

$$X[7] = X^*[1] = \frac{1}{\sqrt{2}}(c[1]-jc[5])$$

Parseval's Relation for the c[k]'s:

$$\sum_{k=0}^{N-1} c^2[k] \ = \ \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$$

Derivation: This follows from the original Parseval's Relation and the easy to check fact that

$$\sum_{k=0}^{N-1} c^2[k] \ = \sum_{k=0}^{N-1} |X[k]|^2 .$$

Note for example that

$$|X[1]-\hat{X}[1]|^2 = \left(\mathrm{Re}(X[1]-\mathrm{Re}(\hat{X}[1])\right)^2 + \left(\mathrm{Im}(X[1]-\mathrm{Im}(\hat{X}[1])\right)^2$$

$$= \left(\frac{1}{\sqrt{2}}c[1]-\frac{1}{\sqrt{2}}\hat{c}[1]\right)^2 + \left(\frac{1}{\sqrt{2}}c[5]-\frac{1}{\sqrt{2}}\hat{c}[5]\right)^2$$

$$= \frac{1}{2}(c[1]-\hat{c}[1])^2 + \frac{1}{2}(c[1]-\hat{c}[5])^2$$

Fact 2: (MSE of the Transform Code) The MSE between the input $x[0],...,x[N-1]$ and the output $\hat{x}[0], ..., \hat{x}[N-1]$ of a transform code based on quantizing $c[0],...,c[N-1]$ equals the sum of the MSE's for each $c[k]$, i.e.

$$\mathrm{MSE} = \sum_{k=0}^{N-1} \mathrm{MSE}(k)$$

where $\mathrm{MSE}(k) = \frac{1}{N}\sum_{n=0}^{N-1}(c[k]-\hat{c}[k])^2$.

Derivation: Like the original DFT, the modified transform, which produces the $c[k]$'s, has a linearity property and a Parseval's Relation. Thus, the derivation is the same as for Fact 1:

$$\left(c[0],...,c[N-1] - \hat{c}[0], ..., \hat{c}[N-1]\right) \quad \text{is the modified DFT of} \quad \left(x[0],...,x[N-1] - \hat{x}[0], ..., \hat{x}[N-1]\right)$$

Therefore, Parsevals Relation for the $c[k]$'s shows

$$\sum_{k=0}^{N-1}|c[k]-\hat{c}[k]|^2 = \frac{1}{N}\sum_{n=0}^{N-1}\left(x[n]-\hat{x}[n]\right)^2.$$