Ch. 6. Frequency-Domain Analysis of Filters

Outline

- Frequency response of filters
- Response to sinusoids and complex exponentials
- Response to periodic signals
- Response to suddenly applied signals
- Frequency response of interconnected systems
- Filtering of sampled continuous-time signals

Introduction

The previous chapter focused on the **time domain** properties of systems, using input signals consisting of unit-impulse functions, step functions, etc.

In particular, we derived the key input-output relationship for LTI systems:

$$x[n] \to \boxed{\text{LTI with impulse response } h[n]} \to y[n] = x[n] * h[n] \,.$$

This is a time domain relationship. The point of this chapter is to find a **frequency domain** relationship, which is often the basis for filter design.

This means we consider sinusoidal signals.

Example.

$$x[n] = \cos(\frac{\pi}{2}n) \to \boxed{\text{LTI } h[n] = \delta[n] - \delta[n-1]} \to y[n] = ?$$

(given in class)

Instead of generalizing further with sinusoids, we back up and consider complex exponential signals instead, and later analyze sinusoids by combining two complex exponential signals.

Frequency response

The response of an LTI system with impulse response $h[n]$ to a complex exponential input signal with frequency $\hat{\omega}$ is the following.

$$x[n] = A e^{\jmath\,\phi} e^{\jmath\,\hat{\omega} n} \to \boxed{\text{LTI } h[n]} \to y[n] = ?.$$

Applying the convolution sum:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k] = \sum_{k=-\infty}^{\infty} A e^{\jmath\,\phi} e^{\jmath\,\hat{\omega}(n-k)} h[k] = \left(\sum_{k=-\infty}^{\infty} h[k]\, e^{-\jmath\,\hat{\omega} k} \right) A e^{\jmath\,\phi} e^{\jmath\,\hat{\omega} n}.$$

So the output signal $y[n]$ turns out to be the input signal scaled by a the complex value given by the summation in the parentheses. This summation is so important that it is given a name: the **frequency response** of the system, and its own symbol[1]:

$$\boxed{\mathcal{H}(\hat{\omega}) \triangleq \sum_{k=-\infty}^{\infty} h[k]\, e^{-\jmath\,\hat{\omega} k}.}$$

Note that the "ordinary" H will be used in Chapter 8 for something related but different.

The frequency response of a system is a *function* of frequency ω, because different frequency components are affected differently by filters; some components are amplified, others attenuated, etc. The frequency response summarizes everything that happens to a complex exponential input signal of any given frequency.

[1] For FIR filters, the sum only has a finite number of nonzero terms, so it is always well defined. For IIR filters, the frequency response is only well defined if the system is **stable**, *i.e.*, if the frequency response is absolutely summable.

Example. Determine the response of the **two-point moving average filter** to a complex exponential signal with frequency $\hat{\omega}$.

$$x[n] = \mathrm{e}^{\jmath \hat{\omega} n} \to \boxed{h[n] = \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-1]} \to y[n] = ?.$$

The frequency response is given by

$$\mathcal{H}(\hat{\omega}) = \sum_{k=-\infty}^{\infty} h[k]\,\mathrm{e}^{-\jmath \hat{\omega} k} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\delta[k] + \frac{1}{2}\delta[k-1]\right)\mathrm{e}^{-\jmath \hat{\omega} k} = \frac{1}{2} + \frac{1}{2}\mathrm{e}^{-\jmath \hat{\omega}}.$$

So for this system

$$x[n] = \mathrm{e}^{\jmath \hat{\omega} n} \xrightarrow{\mathcal{T}} y[n] = \left(\frac{1}{2} + \frac{1}{2}\mathrm{e}^{-\jmath \hat{\omega}}\right)\mathrm{e}^{\jmath \hat{\omega} n}.$$

How do we get insight into what this means? By displaying the frequency response *graphically*.

Since $\mathcal{H}(\hat{\omega})$ is a complex value for any given frequency $\hat{\omega}$, we usually express $\mathcal{H}(\hat{\omega})$ in terms of its magnitude and phase:

$$\mathcal{H}(\hat{\omega}) = |\mathcal{H}(\hat{\omega})|\,\mathrm{e}^{\jmath \angle \mathcal{H}(\hat{\omega})}.$$

- $|\mathcal{H}(\hat{\omega})| = \sqrt{\mathrm{Re}\{\mathcal{H}(\hat{\omega})\}^2 + \mathrm{Im}\{\mathcal{H}(\hat{\omega})\}^2}$ is called the **magnitude response** of the system.
- $\angle \mathcal{H}(\hat{\omega})$ is called the **phase response** of the system.

Usually we plot these two quantities over the range $-\pi$ to π

This form allows a concise input-output relationship for complex exponential signals:

$$\boxed{x[n] = A\mathrm{e}^{\jmath \phi}\mathrm{e}^{\jmath \hat{\omega} n} \xrightarrow{\mathcal{T}} y[n] = \underbrace{|\mathcal{H}(\hat{\omega})|\,A}_{\text{new amplitude}} \; \underbrace{\mathrm{e}^{\jmath(\angle \mathcal{H}(\hat{\omega}) + \phi)}}_{\text{new phase}} \; \underbrace{\mathrm{e}^{\jmath \hat{\omega} n}}_{\text{input exponential signal}}.}$$

For the example above, we have

$$\mathcal{H}(\hat{\omega}) = \frac{1}{2} + \frac{1}{2}\mathrm{e}^{-\jmath \hat{\omega}} = \frac{1}{2}(1 + \cos\hat{\omega} - \jmath \sin\hat{\omega}) = \frac{1+\cos\hat{\omega}}{2} + \jmath\,\frac{-\sin\hat{\omega}}{2}$$

so

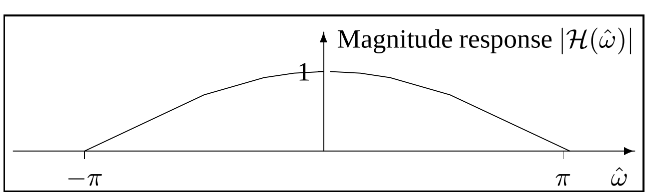
$$|\mathcal{H}(\hat{\omega})| = \sqrt{\left(\frac{1+\cos\hat{\omega}}{2}\right)^2 + \left(\frac{-\sin\hat{\omega}}{2}\right)^2} = |\cos(\hat{\omega}/2)|$$

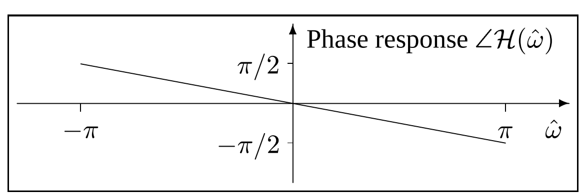
$$\angle \mathcal{H}(\hat{\omega}) = \tan^{-1}\left(\frac{-\sin\hat{\omega}}{1+\cos\hat{\omega}}\right) = \begin{cases} -\hat{\omega}/2, & |\hat{\omega}| \le \pi \\ \text{periodic}, & \text{otherwise.} \end{cases}$$

Alternative derivation using "phase splitting" trick:

$$\mathcal{H}(\hat{\omega}) = \frac{1}{2} + \frac{1}{2}\mathrm{e}^{-\jmath \hat{\omega}} = \mathrm{e}^{-\jmath \hat{\omega}/2}\frac{1}{2}\left[\mathrm{e}^{\jmath \hat{\omega}/2} + \mathrm{e}^{-\jmath \hat{\omega}/2}\right] = \mathrm{e}^{-\jmath \hat{\omega}/2}\cos(\hat{\omega}/2).$$

So the magnitude response and the frequency response have the following graphs.





What can we learn from these pictures?

For example, if the input is a constant signal, such as $x[n] = 5 = 5\mathrm{e}^{\jmath 0n}$, then the output signal is $y[n] = 5\mathcal{H}(0)\,\mathrm{e}^{\jmath 0n} = 5$.

If the input is $x[n] = (-1)^n = \mathrm{e}^{\jmath \pi n}$, then the output signal is $y[n] = \mathcal{H}(\pi)\,\mathrm{e}^{\jmath \pi n} = 0$.

So this system completely removes a sinusoidal frequency component with frequency $\hat{\omega} = \pi$.

Whether removing this frequency component is important or not depends on the application.

Ideal magnitude responses

Here are examples of magnitude responses that are often needed in practical applications. ***(Picture)***

- lowpass filter
- highpass filter
- bandpass filter
- bandstop filter
- notch filter
- resonator

(We focus on the magnitude response here, but in some applications the phase response is also important.)

These are called **ideal** frequency response functions. But in practice we cannot design filters that have exactly these frequency responses so we make compromises.

What type of filter is the two-point moving average? It is a crude lowpass filter. Far from ideal!

Would it be a good filter for removing high frequency noise? Not very!

Example. The first difference filter: $h[n] = \delta[n] - \delta[n-1]$, has frequency response

$$\mathcal{H}(\hat{\omega}) = 1 - \mathrm{e}^{-\jmath\hat{\omega}} = 2\jmath\,\mathrm{e}^{-\jmath\hat{\omega}/2}\left(\frac{\mathrm{e}^{\jmath\hat{\omega}/2} - \mathrm{e}^{-\jmath\hat{\omega}/2}}{2\jmath}\right) = 2\sin(\hat{\omega}/2)\mathrm{e}^{\jmath(\pi/2-\hat{\omega}/2)} \textbf{\textit{ (Picture)}}\,.$$

Properties of frequency response

Periodicity

$\mathcal{H}(\hat{\omega})$ is periodic with period 2π:

$$\mathcal{H}(\hat{\omega}+2\pi) = \mathcal{H}(\hat{\omega})\,.$$

This is natural because digital frequencies $\hat{\omega}$ and $\hat{\omega}+2\pi$ are **equivalent frequencies**.

Proof:

$$\mathcal{H}(\hat{\omega}+2\pi) = \sum_{k=-\infty}^{\infty} h[k]\,\mathrm{e}^{\jmath(\hat{\omega}+2\pi)k} = \sum_{k=-\infty}^{\infty} h[k]\,\mathrm{e}^{\jmath\hat{\omega}k} = \mathcal{H}(\hat{\omega})\,.$$

Conjugate symmetry

$$h[n] \text{ real} \;\Rightarrow\; \mathcal{H}(-\hat{\omega}) = \mathcal{H}^*(\hat{\omega})$$

Proof:

$$\mathcal{H}^*(\hat{\omega}) = \left(\sum_{k=-\infty}^{\infty} h[k]\,\mathrm{e}^{\jmath\hat{\omega}k}\right)^{\star} = \sum_{k=-\infty}^{\infty} h^*[k]\,\mathrm{e}^{-\jmath\hat{\omega}k} = \sum_{k=-\infty}^{\infty} h[k]\,\mathrm{e}^{\jmath(-\hat{\omega})k} = \mathcal{H}(-\hat{\omega})\,.$$

In particular:

- The magnitude response is even

$$|\mathcal{H}(-\hat{\omega})| = |\mathcal{H}(\hat{\omega})|\,.$$

- The phase response is odd

$$\angle\mathcal{H}(-\hat{\omega}) = -\angle\mathcal{H}(\hat{\omega})\,.$$

These facts make sense intuitively since there is “nothing new” in the negative frequencies.

We could restrict attention to $[0,\pi]$ but I will continue to show $[-\pi,\pi]$.

Sinusoidal input signals

Now we turn to sinusoidal signals, which are more important in practice than complex exponential signals and show the following **sine in, sine out** property:

$$\boxed{x[n] = A\cos(\hat{\omega}n + \phi) \to \boxed{\text{LTI } h[n]} \to y[n] = |\mathcal{H}(\hat{\omega})|\, A\cos(\hat{\omega}n + \phi + \angle\mathcal{H}(\hat{\omega})).}$$

Derivation. Do we resort to convolution again? No!

Using inverse Euler:

$$x[n] = \frac{1}{2}A\mathrm{e}^{\jmath\,\phi}\mathrm{e}^{\jmath\,\hat{\omega}n} + \frac{1}{2}A\mathrm{e}^{-\jmath\,\phi}\mathrm{e}^{-\jmath\,\hat{\omega}n}$$

so applying the earlier I/O relation for complex exponential along with the linearity of the system (superposition property) we have:

$$\begin{aligned}
x[n] \xrightarrow{\mathcal{T}} y[n] &= \frac{1}{2}A\,|\mathcal{H}(\hat{\omega})|\,\mathrm{e}^{\jmath\,(\phi+\angle\mathcal{H}(\hat{\omega}))}\mathrm{e}^{\jmath\,\hat{\omega}n} + \frac{1}{2}A\,|\mathcal{H}(-\hat{\omega})|\,\mathrm{e}^{\jmath\,(-\phi+\angle\mathcal{H}(-\hat{\omega}))}\mathrm{e}^{-\jmath\,\hat{\omega}n} \\
&= \frac{1}{2}A\,|\mathcal{H}(\hat{\omega})|\,\mathrm{e}^{\jmath\,(\phi+\angle\mathcal{H}(\hat{\omega}))}\mathrm{e}^{\jmath\,\hat{\omega}n} + \frac{1}{2}A\,|\mathcal{H}(\hat{\omega})|\,\mathrm{e}^{-\jmath\,(\phi+\angle\mathcal{H}(\hat{\omega}))}\mathrm{e}^{-\jmath\,\hat{\omega}n} \\
&= |\mathcal{H}(\hat{\omega})|\,A\cos(\hat{\omega}n + \phi + \angle\mathcal{H}(\hat{\omega})),
\end{aligned}$$

where we used the even symmetry of $|\mathcal{H}(\hat{\omega})|$ and the odd symmetry of $\angle\mathcal{H}(\hat{\omega})$.

Summary: sinusoid in $\Rightarrow$ sinusoid out (with different magnitude and phase)

Sums of sinusoids

Applying linearity:

$$\boxed{\sum_k A_k \cos(\hat{\omega}_k n + \phi_k) \rightarrow \boxed{\text{LTI } \mathcal{H}(\hat{\omega})} \rightarrow \sum_k |\mathcal{H}(\hat{\omega}_k)|\, A_k \cos(\hat{\omega}_k n + \phi_k + \angle\mathcal{H}(\hat{\omega}_k)).}$$

Each **sinusoidal component** in the input signal will appear as a sinusoidal component in the output signal with the *same frequency* but with amplitudes and phases that are affected by the frequency response of the filter. Some frequency components might be amplified, whereas others might be attenuated; some might even be eliminated completely by the filter.

Example.

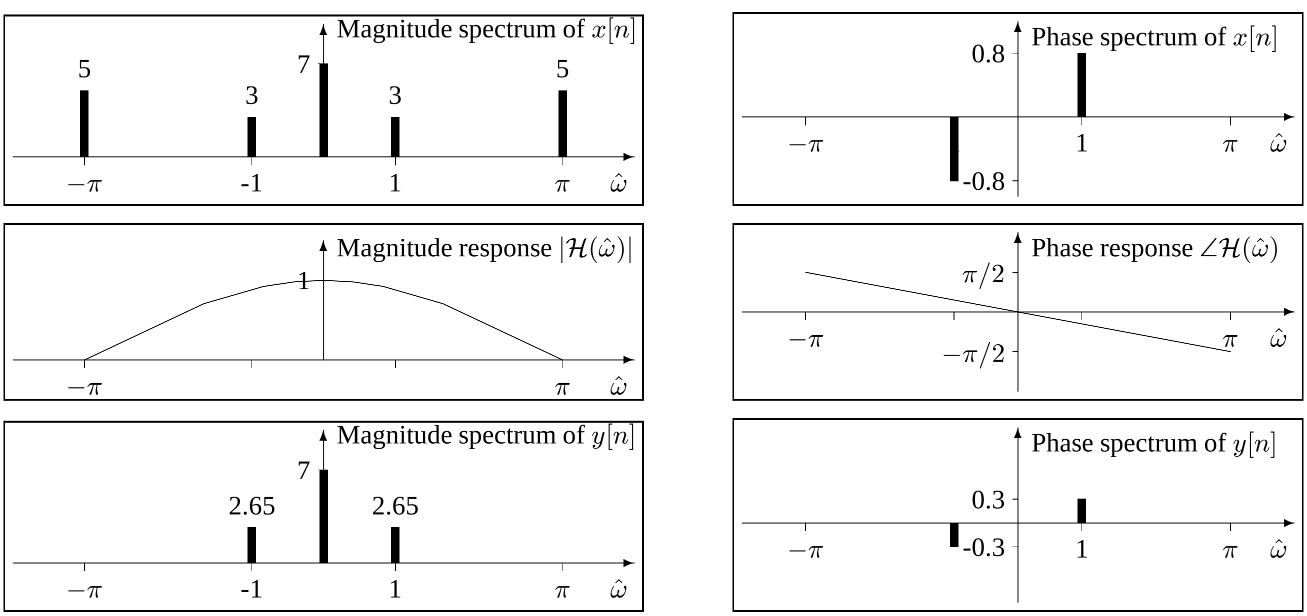
Determine $y[n]$ when $x[n] \rightarrow \boxed{h[n] = \frac{1}{2}\,\delta[n] + \frac{1}{2}\,\delta[n-1]} \rightarrow y[n]$, where $x[n] = 7 + 6\cos(n + 0.8) + 10\cos(\pi n)$.

Recall that the **frequency response** of this two-point **moving average** system is given by $\mathcal{H}(\hat{\omega}) = \cos(\hat{\omega}/2)\mathrm{e}^{-\jmath\hat{\omega}/2}$.

Without any convolution we find the formula as follows:

$$\begin{aligned} y[n] &= \mathcal{H}(0)\,7 + |\mathcal{H}(1)|\,6\cos(n + 0.8 + \angle\mathcal{H}(1)) + |\mathcal{H}(\pi)|\,10\cos(\pi n + \angle\mathcal{H}(\pi)) \\ &= 7 + \cos(\frac{1}{2})6\cos(n + 0.8 - 1/2) \approx 7 + 5.3\cos(n + 0.3). \end{aligned}$$

Although manipulating such formulas is important for problem solving, understanding the concept *graphically* is also very important.



We *multiply* the magnitude of each signal component by the corresponding magnitude response value $|\mathcal{H}(\hat{\omega})|$,
and we *add* the phase of each signal component by the corresponding phase response value $\angle\mathcal{H}(\hat{\omega})$.

Filtering periodic signals

We have considered input signals that are **complex exponentials**, **sinusoids**, and **sums-of-sinusoids**.

We now consider **periodic** input signals, which are an important special case of sums-of-sinusoids.

Suppose we have an LTI system with frequency response $\mathcal{H}(\hat{\omega})$ and the input signal $x[n]$ is N-periodic. Since the system is time invariant,

$$x[n] \xrightarrow{\mathcal{T}} y[n] \;\Rightarrow\; x[n-N] \xrightarrow{\mathcal{T}} y[n-N]\,.$$

But since $x[n]$ is N-periodic, we have $x[n] = x[n-N]$ so it must also be the case that $y[n] = y[n-N]$. So the output signal is also N-periodic.

$$\boxed{\text{For an LTI system: } N\text{-periodic in } \Rightarrow\; N\text{-periodic out.}}$$

- That was a time-domain discussion. If we wanted to determine the response $y[n]$ to a particular input signal $x[n]$ using a time-domain approach, we would have to perform convolution: $y[n] = x[n] * h[n]$. For a periodic input signal $x[n]$, this convolution would often be quite cumbersome.
- Instead, we now consider a frequency-domain perspective, which will greatly simplify finding the output signal $y[n]$ when the input signal $x[n]$ is periodic.

When $x[n]$ and $y[n]$ are N-periodic, we can use the DFT to express $x[n]$ and $y[n]$ as sums-of-complex-exponentials:

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N} kn} \\ y[n] &= \sum_{k=0}^{N-1} Y[k]\, e^{\jmath \frac{2\pi}{N} kn}. \end{aligned}$$

In addition, using our earlier analysis of what happens when complex exponential signals are passed through LTI systems:

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N} kn} \to \boxed{\text{LTI } \mathcal{H}(\hat{\omega})} \to y[n] = \sum_{k=0}^{N-1} \mathcal{H}\!\left(\frac{2\pi}{N}k\right) X[k]\, e^{\jmath \frac{2\pi}{N} kn}.$$

Comparing the preceding two expressions for $y[n]$, and recalling that any signal has a unique DFT, we see that we have shown the following *purely frequency domain* relationship:

$$\boxed{Y[k] = \mathcal{H}\!\left(\frac{2\pi}{N}k\right) X[k]\,.}$$

Each frequency component of the input signal $x[n]$ appears in the output signal with the same frequency $\frac{2\pi}{N}k$ but with its complex amplitude scaled by the corresponding frequency response $\mathcal{H}\left(\frac{2\pi}{N}k\right)$.

As a side comment, notice that if $h[n]$ is supported on $0, \ldots, N-1$, then

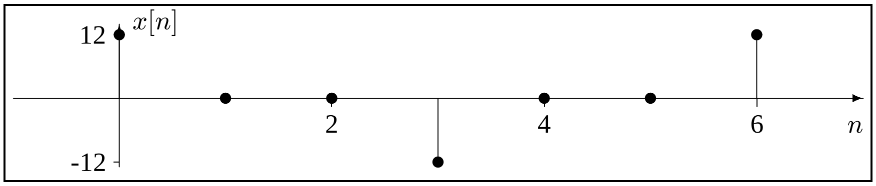
$$\mathcal{H}\!\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-\jmath \frac{2\pi}{N} kn} = N \frac{1}{N} \sum_{n=0}^{N-1} h[n]\, e^{-\jmath \frac{2\pi}{N} kn} = N H[k]\,,$$

where $H[k]$ denotes the N-point DFT of $h[n]$. So there is a relationship between what we are discussing here and the DFT filtering approach described earlier.

If M is small, direct (time-domain) filtering approach is much fast than a DFT approach, even with FFT.

Example.

Considering the following 6-periodic input signal.

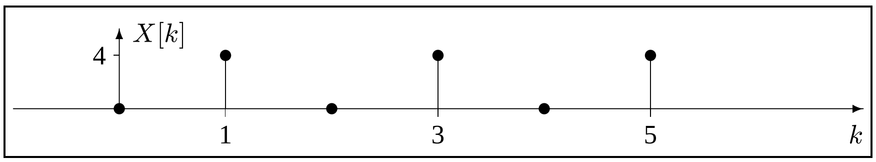


What is the response of the two-point moving average to this input signal?

(In this case the time domain approach would not be so cumbersome, but we illustrate the frequency-domain approach anyway.)

First we find the spectrum of $x[n]$.

$$X[k] = \frac{1}{6}\sum_{n=0}^{5} x[n]\, \mathrm{e}^{-\jmath \frac{2\pi}{6} kn} = \frac{1}{6}[12 - 12\mathrm{e}^{-\frac{2\pi}{6}k3}] = 2 - 2(-1)^k.$$



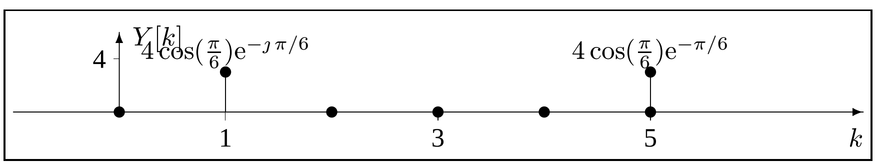
So the input signal has the following expression in terms of sinusoids:

$$x[n] = 8\cos(\frac{\pi}{3}n) + 3\cos(\pi n).$$

One-sided magnitude spectrum $|\mathcal{H}(\hat{\omega})| = |\cos(\omega/2)|$. ***(Picture)***

One-sided phase spectrum $\angle\mathcal{H}(\hat{\omega}) = \begin{cases} -\hat{\omega}/2, & |\hat{\omega}| \leq \pi \\ \ldots, & \text{else.} \end{cases}$ ***(Picture)***

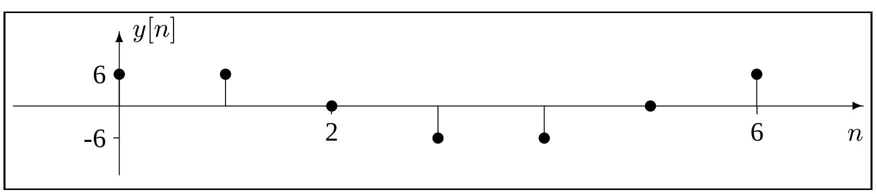
Using $Y[k] = \mathcal{H}\left(\frac{2\pi}{6}k\right)$, the output signal spectrum is as follows.



So the output signal is

$$y[n] = 4\sqrt{3}\cos(\frac{2\pi}{6}n - \frac{\pi}{6}).$$

(We could have found this by using “sine in / sine out” as well.) Here is what the output signal looks like.



Here, the DFT is for *analysis* of the signals, not for the filtering itself.

In this case, time-domain convolution would have been easy enough, but the frequency-domain approach is often more insightful.

6.3 Suddenly applied sinusoidal signals / transient response

We have seen that when an *eternal* sinusoidal signal is the input to an LTI system, the output is also an *eternal* sinusoidal signal with the same frequency:

$$x[n] = A\cos(\hat{\omega}n + \phi) \to \boxed{\text{LTI } \mathcal{H}(\hat{\omega})} \to y[n] = |\mathcal{H}(\hat{\omega})|\, A\cos(\hat{\omega}n + \phi + \angle\mathcal{H}(\hat{\omega})).$$

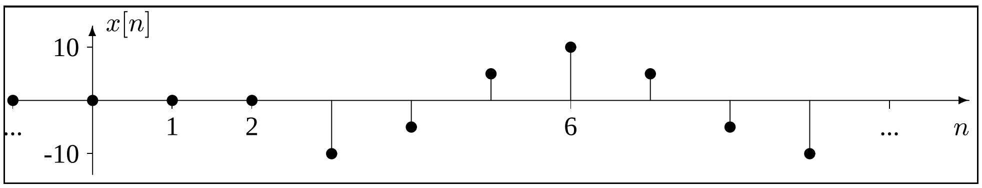
But in practical situations, a "sinusoidal signal" will not be eternal. A more realistic model would be to consider the input signal to be zero until some time n_0, and then it begins to oscillate:

$$x[n] = \begin{cases} 0, & n < n_0 \\ A\cos(\hat{\omega}n + \phi), & n \geq n_0 \end{cases} = A\cos(\hat{\omega}n + \phi)u[n - n_0]\,.$$

The step function is a convenient notation (compared to braces!) for a signal that "switches on" at time n_0.

This type of signal is called a **suddenly applied sinusoidal signal**.

Example. $x[n] = 10\cos(\frac{\pi}{3}n)u[n-3]$



What is the output of an LTI system if the input signal is a suddenly applied sinusoid?

The answer is *not* simply $y[n] = |\mathcal{H}(\hat{\omega})|\, A\cos(\hat{\omega}n + \phi + \angle\mathcal{H}(\hat{\omega}))u[n - n_0]\,.$
We cannot just multiply the input and the output by $u[n - n_0]$. Linearity only allows us to multiply by *constants*, not by signals!

Transient response of FIR filters

To answer this question, we focus on FIR filters. To determine the response of an FIR filter to a suddenly applied sinusoid, we temporarily return to the time-domain input-output relationship for FIR filters:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]\,.$$

For simplicity, we consider a sinusoid that is applied at $n_0 = 0$, *i.e.*,

$$x[n] = A\cos(\hat{\omega}n + \phi)u[n]\,.$$

For this input signal, the output is given by

$$y[n] = \sum_{k=0}^{M} b_k A\cos(\hat{\omega}(n-k) + \phi)u[n-k]\,.$$

For $n < 0$, the output signal is zero, since the system is causal. For $n \geq M$, since $0 \leq k \leq M$, we have $u[n-k] = 1$, so the output signal is given by

$$y[n] = \sum_{k=0}^{M} b_k A\cos(\hat{\omega}(n-k) + \phi) = h[n] * x[n] = |\mathcal{H}(\hat{\omega})|\, A\cos(\hat{\omega}n + \phi + \angle\mathcal{H}(\hat{\omega})), \qquad n \geq M.$$

This is called the **steady-state response** of the system.
For $0 \leq n < M$, $u[n-k]$ is zero when $k > n$, so the response is

$$y[n] = \sum_{k=0}^{n} b_k A\cos(\hat{\omega}(n-k) + \phi), \qquad 0 \leq n < M.$$

This is called the **transient response** of the system.

In summary, for a sinusoidal signal applied suddenly at time $n_0 = 0$, the response of an Mth-order FIR filter with frequency response $\mathcal{H}(\hat{\omega})$ is given by:

$$y[n] = \begin{cases} 0, & n < 0 \\ \sum_{k=0}^{n} b_k A \cos(\hat{\omega}(n-k) + \phi), & 0 \le n < M \\ |\mathcal{H}(\hat{\omega})|\, A \cos(\hat{\omega} n + \phi + \angle\mathcal{H}(\hat{\omega})), & n \ge M, \end{cases}$$

where $\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{M} b_k \mathrm{e}^{-\jmath \hat{\omega} k}$.

Practically speaking, we perform the following analyses.
- To find the **transient response**, we use the time-domain formula.
- To find the **steady-state response**, we use the frequency-domain formula.

Example. Suppose the input signal $x[n] = 3u[n] + 8\cos(\frac{\pi}{4}n)u[n]$ is the input to the 2nd-order FIR filter with $b_k = \left\{1, -\sqrt{2}, 1\right\}$. Determine the output signal.

This system is causal, so the output is zero for $n < 0$.

Next we find the steady-state response. The frequency response is $\mathcal{H}(\hat{\omega}) = 1 - \sqrt{2}\mathrm{e}^{-\jmath \hat{\omega}} + \mathrm{e}^{-\jmath 2\hat{\omega}}$,
so $\mathcal{H}(0) = 1 - \sqrt{2} + 1 = 2 - \sqrt{2} \approx 0.6$ and $\mathcal{H}(\pi/4) = 1 - \sqrt{2}\mathrm{e}^{-\jmath \pi/4} + \mathrm{e}^{\jmath \pi/2} = 0$.
Thus, the steady-state response, for $n \ge M = 2$ is $y[n] = 3(2 - \sqrt{2}) \approx 1.8$.

For the transient response, we apply the time-domain formula to see:

$$\begin{aligned} y[0] &= b_0 x[0] = 11 \\ y[1] &= b_0 x[1] + b_1 x[0] = 1 \cdot (3 + 4\sqrt{2}) + (-\sqrt{2}) \cdot (3 + 0) = 3 + \sqrt{2} \approx 4.4. \end{aligned}$$

Rather than using braces, the most concise expression for $y[n]$ is the following:

$$y[n] = \underbrace{11\delta[n] + (3+\sqrt{2})\delta[n-1]}_{\text{transient response}} + \underbrace{3(2-\sqrt{2})u[n-2]}_{\text{steady-state response}}.$$

Example. Suppose the input signal $x[n] = 3u[n] + 8\cos(\frac{\pi}{2}n)u[n]$ is the input to the 2nd-order FIR filter with $b_k = \{1, 2, 1\}$. Determine the output signal.

This system is causal, so the output is zero for $n < 0$.

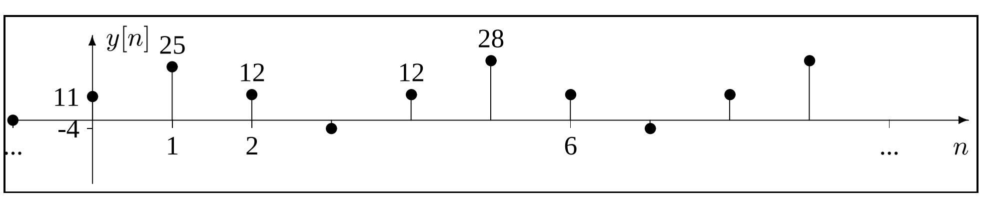
Next we find the steady-state response. The frequency response is $\mathcal{H}(\hat{\omega}) = 1 + 2\mathrm{e}^{-\jmath \hat{\omega}} + \mathrm{e}^{-\jmath 2\hat{\omega}}$,
so $\mathcal{H}(0) = 1 + 2 + 1 = 4$ and $\mathcal{H}(\pi/2) = 1 + 2\mathrm{e}^{-\jmath \pi/2} + \mathrm{e}^{\jmath \pi} = 2\mathrm{e}^{-\jmath \pi/2}$.
Thus, the steady-state response, for $n \ge M = 2$ is $y[n] = 4 \cdot 3 + 2 \cdot 8\cos(\frac{\pi}{2}n - \frac{\pi}{2}) = 12 + 16\cos(\frac{\pi}{2}n - \frac{\pi}{2})$.

For the transient response, we apply the time-domain formula to see:

$$\begin{aligned} y[0] &= b_0 x[0] = 11 \\ y[1] &= b_0 x[1] + b_1 x[0] = 1 \cdot (3 + 0) + 2 \cdot 11 = 25. \end{aligned}$$

Rather than using braces, the most concise expression for $y[n]$ is the following:

$$y[n] = \underbrace{11\delta[n] + 25\delta[n-1]}_{\text{transient response}} + \underbrace{\left(12 + 16\cos(\frac{\pi}{2}n - \frac{\pi}{2})\right) u[n-2]}_{\text{steady-state response}}.$$



6.6

Interconnected LTI systems

We previously analyzed the interconnection of LTI systems in the time domain.

- Series connection of two LTI systems yields an overall impulse response of $h[n] = h_1[n] * h_2[n]$.
- Parallel connection of two LTI systems yields an overall impulse response of $h[n] = h_1[n] + h_2[n]$.

We now analyze such interconnections in the **frequency domain**.

For two LTI systems connected in series (aka **cascade**):

$$x[n] = \mathrm{e}^{\jmath \hat{\omega} n} \to \boxed{\mathcal{H}_1(\hat{\omega})} \overset{w[n]=\mathcal{H}_1(\hat{\omega})\mathrm{e}^{\jmath \hat{\omega} n}}{\to} \boxed{\mathcal{H}_2(\hat{\omega})} \to y[n] = \mathcal{H}_2(\hat{\omega})\, \mathcal{H}_1(\hat{\omega})\, \mathrm{e}^{\jmath \hat{\omega} n}.$$

So the overall frequency response of the interconnected systems is the *product* of the two frequency responses

$$x[n] \to \boxed{\mathcal{H}(\hat{\omega}) = \mathcal{H}_1(\hat{\omega})\, \mathcal{H}_2(\hat{\omega})} \to y[n]\,.$$

We see therefore a correspondence between convolution in the time domain and multiplication in the frequency domain.

Time Domain		Frequency Domain
$h_1[n] * h_2[n]$	$\Longleftrightarrow$	$\mathcal{H}_1(\hat{\omega})\, \mathcal{H}_2(\hat{\omega})$
Convolution		Multiplication

For two LTI systems connected in **parallel** ***(Picture)*** , the overall frequency response is the *sum* of the two frequency responses:

$$\mathcal{H}(\hat{\omega}) = \mathcal{H}_1(\hat{\omega}) + \mathcal{H}_2(\hat{\omega})\,.$$

Time Domain		Frequency Domain
$h_1[n] + h_2[n]$	$\Longleftrightarrow$	$\mathcal{H}_1(\hat{\omega}) + \mathcal{H}_2(\hat{\omega})$
Addition		Addition

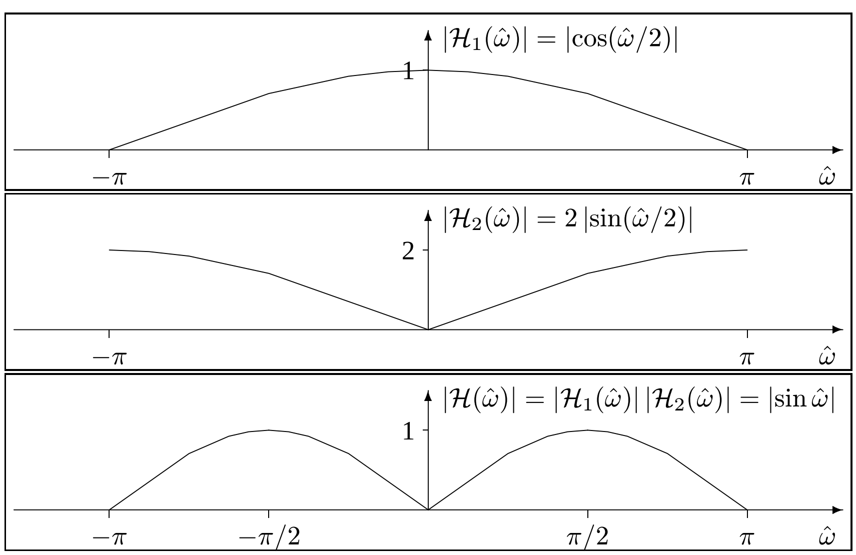
Example. Find the overall frequency response of the following cascade:

$$x[n] \to \boxed{\text{Lowpass: } h_1[n] = \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-1]} \to \boxed{\text{Highpass: } h_2[n] = \delta[n] - \delta[n-1]} \to y[n]\,.$$

The overall frequency response is

$$\mathcal{H}(\hat{\omega}) = \mathcal{H}_1(\hat{\omega})\, \mathcal{H}_2(\hat{\omega}) = \left(\frac{1}{2} + \frac{1}{2}\mathrm{e}^{-\jmath \hat{\omega}}\right)\left(1 - \mathrm{e}^{-\jmath \hat{\omega}}\right) = \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-\jmath 2\hat{\omega}} = \mathrm{e}^{-\jmath \hat{\omega}}\, \jmath\, \frac{\mathrm{e}^{\jmath \hat{\omega}} - \mathrm{e}^{-\jmath \hat{\omega}}}{2\jmath} = \mathrm{e}^{\jmath(\pi/2 - \hat{\omega})} \sin \hat{\omega}.$$

In particular, the magnitude responses multiply: $|\mathcal{H}(\hat{\omega})| = |\mathcal{H}_1(\hat{\omega})|\, |\mathcal{H}_2(\hat{\omega})| = |\cos(\hat{\omega}/2)|\, |2\sin(\hat{\omega}/2)| = |\sin \hat{\omega}|\,.$



Example.

Find the impulse response of the following cascade:

$$x[n] \to \boxed{\mathcal{H}_1(\hat{\omega}) = 1 + 2e^{\jmath\hat{\omega}} - 3e^{\jmath 2\hat{\omega}}} \to \boxed{\mathcal{H}_2(\hat{\omega}) = \frac{1}{1 - e^{\jmath\hat{\omega}}}} \to y[n]\,.$$

The overall frequency response is

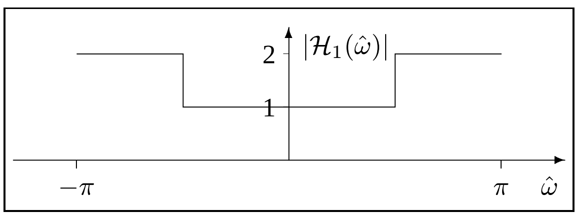
$$\mathcal{H}(\hat{\omega}) = \mathcal{H}_2(\hat{\omega})\,\mathcal{H}_1(\hat{\omega}) = \left(1 + 2e^{\jmath\hat{\omega}} - 3e^{\jmath 2\hat{\omega}}\right)\frac{1}{1 - e^{\jmath\hat{\omega}}} = \left(1 - e^{\jmath\hat{\omega}}\right)\left(1 - 3e^{\jmath 2\hat{\omega}}\right)\frac{1}{1 - e^{\jmath\hat{\omega}}} = 1 - 3e^{\jmath 2\hat{\omega}}.$$

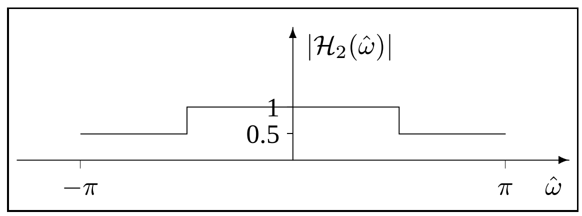
So by **coefficient matching**, the impulse response of this (noncausal) system is

$$h[n] = -3\delta[n+2] + \delta[n]\,.$$

Example. Dolby noise reduction (simplified).

$$x \to \boxed{\text{high freq. boost } H_1} \to \underset{\substack{\uparrow \\ \text{noise}}}{\bigoplus} \to \boxed{\text{high freq. reduce } H_2} \to y = x + \text{reduced high frequency noise.}$$





Filter design preliminaries

Example. (An allpass filter.)

What filter amplifies all frequency components equally with gain = 10?

Apparently we want $|\mathcal{H}(\hat{\omega})| = 10$ and $\angle\mathcal{H}(\hat{\omega}) = 0$. ***(Picture)***

So we want $\mathcal{H}(\hat{\omega}) = 10$.

What is the corresponding impulse response $h[n]$?

(Notice how now we are *first* specifying $\mathcal{H}(\hat{\omega})$ and then determining $h[n]$.)

$$\mathcal{H}(\hat{\omega}) = \sum_{k=-\infty}^{\infty} h[k]\,\mathrm{e}^{-\jmath\hat{\omega}k} = \cdots + h[-1]\,\mathrm{e}^{\jmath\hat{\omega}} + h[0] + h[1]\,\mathrm{e}^{-\jmath\hat{\omega}} + h[2]\,\mathrm{e}^{-\jmath\hat{\omega}2} + \cdots.$$

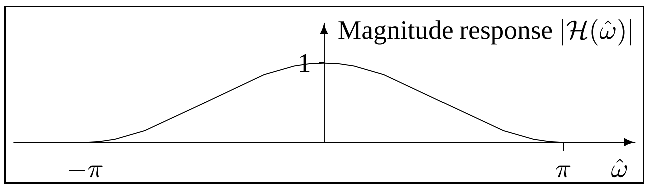
So by **coefficient matching**:

$$h[n] = \begin{cases} 10, & n = 0 \\ 0, & \text{otherwise} \end{cases} = 10\delta[n].$$

Example. (A lowpass filter.)

Since the two-point moving average was a fairly poor lowpass filter, let us try to design a somewhat better one.

Acknowledging that the ideal lowpass is unachievable, let us try for the following magnitude response.



where $|\mathcal{H}(\hat{\omega})| = \frac{1}{2} + \frac{1}{2}\cos\hat{\omega}$.

What phase response should we choose? For simplicity, we start with $\angle\mathcal{H}(\hat{\omega}) = 0$, so

$$\mathcal{H}(\hat{\omega}) = |\mathcal{H}(\hat{\omega})|\,\mathrm{e}^{\jmath\angle\mathcal{H}(\hat{\omega})} = \frac{1}{2} + \frac{1}{2}\cos\hat{\omega} = \underbrace{\frac{1}{2}}_{h[0]} + \underbrace{\frac{1}{4}}_{h[-1]}\,\mathrm{e}^{\jmath\hat{\omega}} + \underbrace{\frac{1}{4}}_{h[1]}\,\mathrm{e}^{-\jmath\hat{\omega}}.$$

By coefficient matching, we see that the corresponding impulse response would be

$$h[n] = \frac{1}{4}\delta[n+1] + \frac{1}{2}\delta[n] + \frac{1}{4}\delta[n-1].$$

Is this a causal filter? No.
How could we make it causal? We could try just shifting it over by one sample:

$$h_s[n] = h[n-1] = \frac{1}{4}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2].$$

Since this is just a "guess" we must compute the frequency response of $h_s[n]$ and see if it has the desired properties.

$$\begin{aligned} \mathcal{H}_s(\hat{\omega}) &= \sum_{k=-\infty}^{\infty} h_s[k]\,\mathrm{e}^{-\jmath\hat{\omega}k} = \frac{1}{4} + \frac{1}{2}\mathrm{e}^{-\jmath\hat{\omega}} + \frac{1}{4}\mathrm{e}^{-\jmath 2\hat{\omega}} \quad \text{now a phase trick:} \\ &= \left[\frac{1}{4}\mathrm{e}^{\jmath\hat{\omega}} + \frac{1}{2} + \frac{1}{4}\mathrm{e}^{\jmath\hat{\omega}}\right]\mathrm{e}^{-\jmath\hat{\omega}} = \left[\frac{1}{2} + \frac{1}{2}\cos\hat{\omega}\right]\mathrm{e}^{-\jmath\hat{\omega}} = |\mathcal{H}(\hat{\omega})|\,\mathrm{e}^{-\jmath\hat{\omega}}. \end{aligned}$$

So $|\mathcal{H}_s(\hat{\omega})| = \frac{1}{2} + \frac{1}{2}\cos\hat{\omega}$ and $\angle\mathcal{H}_s(\hat{\omega}) = -\hat{\omega}$. ***(Picture)***

So the above magnitude response corresponds to the simple 2nd-order filter with impulse response $h_s[n]$ and coefficients $b_k = \{1/4, 1/2, 1/4\}$.

Example. **Filter design for removing 60Hz hum**

Now we return to the example of trying to eliminate 60 Hz hum, when the sampling frequency is $f_s = 480$ Hz, so the corresponding digital frequency is $\omega_0 = 2\pi \frac{f}{f_s} = \pi/4$.

We would like other frequencies to be relatively unaffected, but it is impossible to achieve that goal perfectly. As a minimal constraint, let us require that $\mathcal{H}(0) = 1$.

Can we do this with a first-order filter? When $h[n] = b_0 + b_1\delta[n-1]$, the frequency response is $\mathcal{H}(\hat{\omega}) = b_0 + b_1 e^{-\jmath\hat{\omega}}$.
Substituting in the two conditions $\mathcal{H}(\pi/4) = 0$ and $\mathcal{H}(0) = 1$ yields the following two equations

$$\begin{aligned} 0 &= b_0 + b_1 e^{-\jmath\pi/4} \\ 1 &= b_0 + b_1 e^{-\jmath 0} = b_0 + b_1. \end{aligned}$$

The solution to these equations gives complex values for the coefficients. We want a real system, so we must consider a higher order system.

A 2nd-order filter has impulse response $h[n] = b_0 + b_1\delta[n-1] + b_2\delta[n-2]$ and frequency response

$$\mathcal{H}(\hat{\omega}) = b_0 + b_1 e^{-\jmath\hat{\omega}} + b_2 e^{-\jmath 2\hat{\omega}}.$$

Substituting in the two conditions $\mathcal{H}(\pi/4) = 0$ and $\mathcal{H}(0) = 1$ yields the following two equations

$$\begin{aligned} 0 &= b_0 + b_1 e^{-\jmath\pi/4} + b_2 e^{-\jmath\pi/2} \\ 1 &= b_0 + b_1 + b_2. \end{aligned}$$

One way to obtain a real solution is to require $b_2 = b_0$ in which case the solution is

$$b_0 = b_2 = \frac{1}{2-\sqrt{2}}, \qquad b_1 = -\frac{\sqrt{2}}{2-\sqrt{2}}.$$

This is the design illustrated earlier in the Part 5 lecture notes.

To truly understand how well this filter works, we should examine its frequency response.

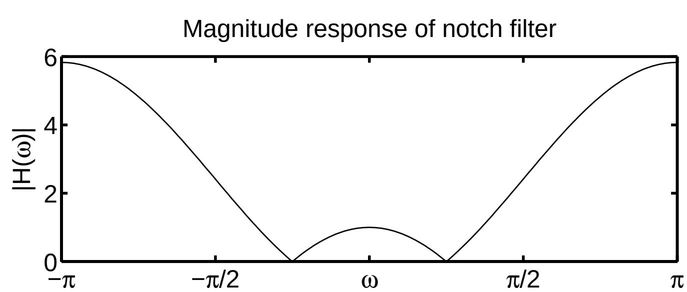
$$\begin{aligned} \mathcal{H}(\hat{\omega}) &= b_0 + b_1 e^{-\jmath\hat{\omega}} + b_0 e^{-\jmath 2\hat{\omega}} = e^{-\jmath\hat{\omega}}\left(b_0 e^{-\hat{\omega}} + b_1 + b_0 e^{-\jmath\hat{\omega}}\right) \\ &= e^{-\jmath\hat{\omega}}\left(b_1 + 2b_0\cos\hat{\omega}\right). \end{aligned}$$

So the magnitude response is

$$|\mathcal{H}(\hat{\omega})| = \left| \frac{2}{2-\sqrt{2}}\cos\hat{\omega} - \frac{\sqrt{2}}{2-\sqrt{2}} \right|.$$

The easiest way to plot this is to use MATLAB's `freqz` command as follows.

```
b = [1 -sqrt(2) 1]/(2-sqrt(2));
om = linspace(-pi,pi,201);
H = freqz(b, [1], om);
clf, subplot(211)
plot(om, abs(H))
```



This "trial and error" approach to filter design did accomplish the goal of having $\mathcal{H}(\pi/4) = 0$ and $\mathcal{H}(0) = 1$, but the large amplification of high frequencies is an undesirable side effect! The next chapter leads to more systematic approaches to filter design using z-transforms.

6.8

Filtering sampled continuous-time signals

One of the most interesting uses of digital signal processing is to apply digital filters to process sampled analog signals. Many audio systems now include such digital processing as an integral component.

So far we have analyzed what happen when a *discrete-time* signal is passed through a discrete-time filter. Before we can apply such filters to sampled analog signals, we must analyze what happens in the following scenario.

$$x(t) \to \boxed{\text{Ideal C-D with } f_s} \to x[n] \to \boxed{\text{LTI } \mathcal{H}(\hat{\omega})} \to y[n] \to \boxed{\text{Ideal D-C with } f_s} \to y(t)$$

Every component in this system is linear, so as usual, we begin our analysis with sinusoidal signals, knowing that later we can consider the more interesting case of sums of sinusoids.

We consider the case where there is no aliasing, so the frequency f_0 of the input signal $0 \le f_0 < f_s/2$.
If $x(t) = \cos(2\pi f_0 t + \phi)$ then
$x[n] = x(nT_s) = \cos(2\pi f_0 n T_s + \phi) = \cos(\omega_0 n + \phi)$ where the digital frequency is $\omega_0 = 2\pi \frac{f_0}{f_s} \in [0, \pi)$.
Thus, applying the frequency response of the LTI system: $y[n] = |\mathcal{H}(\omega_0)| \cos(\omega_0 n + \phi + \angle\mathcal{H}(\omega_0))$.
For the ideal interpolator, the output signal will be

$$y(t) = |\mathcal{H}(\omega_0)| \cos(2\pi f_0 t + \phi + \angle\mathcal{H}(\omega_0)) = \left|\mathcal{H}\left(2\pi \frac{f_0}{f_s}\right)\right| \cos\left(2\pi f_0 t + \phi + \angle\mathcal{H}\left(2\pi \frac{f_0}{f_s}\right)\right).$$

This is essentially another "sine in / sine out" relationship!

Thus, at least as far as sinusoidal input signals are concerned (and more generally for any suitably bandlimited signal), the overall system acts like a filter with frequency response

$$\mathcal{H}\left(2\pi \frac{f_0}{f_s}\right).$$

(In fact, end-to-end this is a LTI system for **bandlimited** continuous-time signals.)

Thanks to this analysis, we can now consider the *design* of digital filters even for (sampled) analog signals.

Example.

$$x(t) = 3\cos(2\pi 60 t) + A\cos(2\pi f_0 t + \phi).$$

The first term is 60Hz hum, contaminating the second signal which is the signal of interest.

Suppose $f_0 = 120$Hz. If the sampling rate is $f_s = 480$Hz and the sampled signal is passed through an LTI system with impulse response

$$h[n] = \frac{1}{2-\sqrt{2}}\delta[n] - \frac{\sqrt{2}}{2-\sqrt{2}}\delta[n-1] + \frac{1}{2-\sqrt{2}}\delta[n-2],$$

for which

$$\mathcal{H}(\hat{\omega}) = e^{-\jmath\hat{\omega}} \left(\frac{2}{2-\sqrt{2}}\cos\hat{\omega} - \frac{\sqrt{2}}{2-\sqrt{2}}\right).$$

Note that $\mathcal{H}\big(2\pi\frac{60}{480}\big) = \mathcal{H}(\pi/4) = 0$ and $\mathcal{H}\big(2\pi\frac{120}{480}\big) = \mathcal{H}(\pi/2) = e^{\jmath\pi/2}\frac{\sqrt{2}}{2-\sqrt{2}} \approx 2.4e^{\jmath\pi/2}$.
Thus, the output signal will be

$$\begin{aligned} y(t) &= \left|\mathcal{H}\left(2\pi\frac{60}{480}\right)\right| 3\cos\left(2\pi 60 t + \angle\mathcal{H}\left(2\pi\frac{60}{480}\right)\right) + \left|\mathcal{H}\left(2\pi\frac{120}{480}\right)\right| A\cos\left(2\pi 120 t + \phi + \angle\mathcal{H}\left(2\pi\frac{120}{480}\right)\right) \\ &= 2.4A\cos(2\pi 120 t + \phi + \pi/2). \end{aligned}$$

The 60Hz component was removed completely, as desired, but the 120Hz component was amplified by 2.4, which is an undesirable side effect. So now it is really truly time to move on to better filter design!