Ch. 7: Z-transform

- Definition
- Properties
  - ○ linearity / superposition
  - ○ time shift
  - ○ convolution: $y[n] = h[n] * x[n] \iff Y(z) = H(z)\,X(z)$
- Inverse z-transform by coefficient matching
- System function $H(z)$
  - ○ poles, zeros, pole-zero plots
  - ○ conjugate pairs
  - ○ relationship to $\mathcal{H}(\hat{\omega})$
- Interconnection of systems
  - ○ Cascade / series connection
  - ○ Parallel connection
  - ○ Feedback connection
- Filter design

Reading

- Text Ch. 7
- (Section 7.9 optional)

$$\boxed{\textbf{z-Transforms}}$$

Introduction

So far we have discussed the time domain (n), and the frequency domain ($\hat{\omega}$). We now turn to the z domain (z).

Why?
- Other types of input signals:
  step functions, geometric series, finite-duration sinusoids.
- Transient analysis.
  Right now we only have a time-domain approach.
- Filter design: approaching a systematic method.
- Analysis and design of IIR filters.

Definition

The (one-sided) **z transform** of a signal $x[n]$ is defined by

$$X(z) = \sum_{k=0}^{\infty} x[k]\, z^{-k} = x[0] + x[1]\, z^{-1} + x[2]\, z^{-2} + \cdots ,$$

where z can be any complex number. This is a function of z.

We write $X(z) = \mathcal{Z}\{x[n]\}$, where $\mathcal{Z}\{\cdot\}$ denotes the z-transform *operation*. We call $x[n]$ and $X(z)$ z-**transform pairs**, denoted

$$x[n] \iff X(z).$$

Left side: function of *time* (n)   Right side: function of z

Example. For the signal

$$x[n] = 3\delta[n] + 7\delta[n-6] = \begin{cases} 3, & n = 0 \\ 7, & n = 6 \\ 0, & \text{otherwise,} \end{cases}$$

the z-transform is

$$X(z) = \sum_k (3\delta[k] + 7\delta[k-6])z^{-k} = 3 + z^{-6}.$$

This is a polynomial in z^{-1}; specifically: $X(z) = 3 + 7(z^{-1})^6$.

$$\begin{array}{ccc} n\text{-Domain} & & z\text{-Domain} \\ x[n] = \sum_k x[k]\,\delta[n-k] & \iff & X(z) = \sum_k x[k]\,z^{-k} \end{array}$$

For causal signals, the z-transform is one-to-one, so using **coefficient matching** we can determine[1] the signal $x[n]$ from its z-transform. We call this "taking the inverse z-transform."

Example. Given

$$Y(z) = 3z^{-1} + 7z^{-5},$$

the corresponding signal is

$$y[n] = 3\delta[n-1] + 7\delta[n-5].$$

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[1] For infinite duration signals there are some technicalities that we leave to EECS 306 and 451.

Properties of the z-transform

These are all easy to show from the definition of the z-transform.

Linearity

$$\alpha x[n] + \beta y[n] \Longleftrightarrow \alpha X(z) + \beta Y(z)$$

Time-shift

$$x[n - n_0] \Longleftrightarrow X(z)\, z^{-n_0}$$

Convolution (later)

$$y[n] = x[n] * h[n] \Longleftrightarrow Y(z) = H(z)\, X(z)$$

The system function of an LTI system

Summary of ways to characterize filters.
- Time domain:
  - I/O relation / diffeq $y[n] = \cdots$
  - Filter coefficients $\{b_k\}$
  - Impulse response $h[n]$
  - Block diagram
- "Frequency" domain:
  - Frequency response $\mathcal{H}(\hat{\omega})$
  - System function $H(z)$
  - Pole-zero plot (z domain poles, zeros, gain)

The z-transform is particularly useful when we consider LTI systems.
The z-transform of the impulse response $h[n]$ of an LTI system is denoted $H(z)$ and is called the **system function**.

We will focus on causal systems, so the system function, also called the **transfer function**, is given by

$$H(z) = \sum_{k=0}^{\infty} h[k]\, z^{-k} = h[0] + h[1]\, z^{-1} + h[2]\, z^{-2} + \cdots .$$

Example. The two-point moving average filter has impulse response $h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$. Its system function is

$$H(z) = \frac{1}{2} + \frac{1}{2}z^{-1}.$$
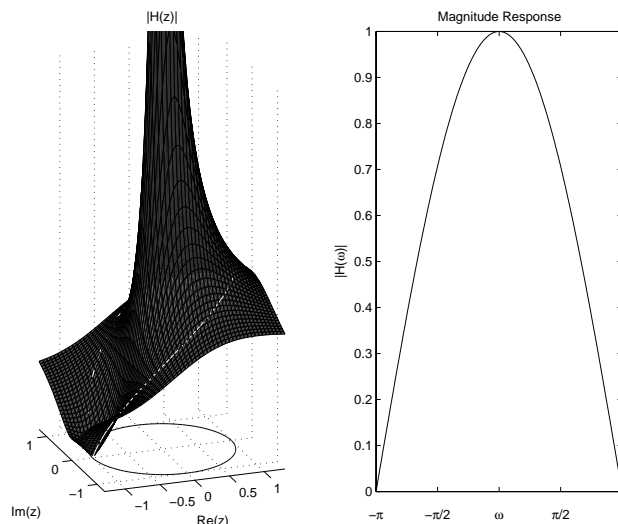
Example. A filter has the system function $H(z) = z^{-3}$. What does it do?

Taking the inverse z-transform by coefficient matching we see

$$h[n] = \delta[n - 3],$$

so this system delays the signal by 3 samples.

This is why a unit delay is often denoted by $\to \boxed{z^{-1}} \to$ in block diagrams.

Example. The z-transform is particularly useful for IIR filters. Suppose $h[n] = a^n u[n]$, for some number a with $|a| < 1$. Then

$$H(z) = \sum_{k=0}^{\infty} h[k] z^{-k} = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} (az^{-1})^k = \frac{1}{1 - az^{-1}}.$$

Does this hold for all values of z? No! We need to have $|az^{-1}| < 1$ *i.e.*, $|z| > |a|$. The subset of the complex plane $\{z \in \mathbb{C} : |z| > |a|\}$ is called the **region of convergence** of the z-transform. More details in 306/451.

Example. A filter has impulse response $h[n] = \delta[n] + (1/2)^n u[n]$. The system function is

$$H(z) = 1 + \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{2 - \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}}.$$

Notice that this is a *ratio of polynomials* in z^{-1}. We call such system functions **rational** system functions, and they are the only kind we will need for designing FIR and IIR filters.

The z domain versus the $\hat{\omega}$ domain

The z domain is a generalization of the frequency domain for causal signals such as suddenly-applied sinusoids[2]. Notice the similarity:
- Frequency response of a causal LTI system: $\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{\infty} h[k] e^{-\jmath\hat{\omega}k}$.
- System function for a causal LTI system: $H(z) = \sum_{k=0}^{\infty} h[k] z^{-k}$.

Comparing, we see that if we replace z with $e^{\jmath\hat{\omega}}$ (which is a complex value after all for any given $\hat{\omega}$) in the system function, then we obtain the frequency response:

$$\mathcal{H}(\hat{\omega}) = H(z)\Big|_{z=e^{\jmath\hat{\omega}}} = H\big(e^{\jmath\hat{\omega}}\big).$$

This is why we use the notation $\mathcal{H}(\hat{\omega})$ and $H(z)$, because they are very closely related, by making the substitution

$$\boxed{z = e^{\jmath\hat{\omega}}.}$$

For such values of z, we have $|z| = |e^{\jmath\hat{\omega}}| = 1$ and $\angle z = \angle e^{\jmath\hat{\omega}} = \hat{\omega}$, so these values of z lie on the **unit circle** in the complex plane. So $H(z)$ is more general because one can also consider values of z that are not on the unit circle, and this flexibility is helpful for filter design.

Furthermore, z^{-1} is easier to write than $e^{-\jmath\hat{\omega}}$!

Example. What is the magnitude response of the filter with system function $H(z) = z^{-3}$??
$\mathcal{H}(\hat{\omega}) = H\big(e^{\jmath\hat{\omega}}\big) = e^{-\jmath 3\hat{\omega}}$, so $|\mathcal{H}(\hat{\omega})| = |e^{-\jmath 3\hat{\omega}}| = 1$. So the 3-sample delay system is an **all pass** filter. The magnitudes of sinusoidal components are unaffected.

Where is DC? At $\hat{\omega} = 0$, so $z = 1$.
Where are high frequencies? Near $\hat{\omega} = \pm\pi$, so $z = -1$.

The frequency interval $-\pi \leq \hat{\omega} \leq \pi$ corresponds to the entire unit circle.

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[2]For eternal periodic signals, the z-transform is undefined, so the frequency domain remains our primary tool for those signals.

Summary for FIR filters

Time domain:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k], \qquad h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

Frequency response ($\hat{\omega}$):

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-\jmath \hat{\omega} k}$$

System function (z):

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}.$$

Avoiding convolution

Time domain (convolution):

$$x[n] \to \boxed{h[n]} \to y[n] = x[n] * h[n]$$

z domain (multiplication!):

$$X(z) \to \boxed{H(z)} \to Y(z) = X(z)\,H(z)$$

Example. (Aperiodic input signal, so $\mathcal{H}(\hat{\omega})$ is not directly useful!)

$$x[n] = 3\delta[n] + \delta[n-2] \to \boxed{\mathcal{H}(\hat{\omega}) = e^{-\jmath\hat{\omega}} - e^{-\jmath 2\hat{\omega}}} \to y[n] = ?$$

To avoid convolution, use z domain. $X(z) = 3 + z^{-2}$, $H(z) = z^{-1} + z^{-2}$, so

$$Y(z) = X(z)\,H(z) = (3 + z^{-2})(z^{-1} + z^{-2}) = 3z^{-1} + 3z^{-2} + z^{-3} + z^{-4}.$$

Taking the inverse z-transform yields

$$y[n] = 3\delta[n-1] + 3\delta[n-2] + \delta[n-3] + \delta[n-4].$$

Fact. **Time-domain convolution of finite-duration signals is equivalent to z domain polynomial multiplication!**
Indeed, in MATLAB, one uses the `conv` command for multiplying polynomials!

Proof of convolution property

$$
\begin{aligned}
Y(z) &= \mathcal{Z}\{y[n]\} = \mathcal{Z}\{x[n] * h[n]\} = \mathcal{Z}\left\{\sum_k h[k]\,x[n-k]\right\} \\
&= \sum_k h[k]\,\mathcal{Z}\{x[n-k]\} \qquad \text{by linearity of } \mathcal{Z}\{\cdot\} \\
&= \sum_k h[k]\,X(z)\,z^{-k} \qquad \text{by time-shift property of } z\text{-transform} \\
&= X(z) \sum_k h[k]\,z^{-k} = X(z)\,H(z).
\end{aligned}
$$

Three "laws" of LTI systems (*cf.* Ohm's law)

Given any two of the three $\{X(z)\,, H(z)\,, Y(z)\}$ we can find the third as follows. All three are useful for different purposes.
• $Y(z) = H(z)\,X(z)$
• $H(z) = Y(z)\,/X(z)$
• $X(z) = Y(z)\,/H(z)$

Finding $H(z)$ from I/O relation We can use the relationship $H(z) = Y(z)/X(z)$ to find $H(z)$ easily.

Example. (Analysis)
A 2-point moving average filter has the input-output relationship

$$y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

so in the z domain, using the linearity and time-shift properties:

$$Y(z) = \frac{1}{2}X(z) + \frac{1}{2}X(z)\,z^{-1} = \left(\frac{1}{2} + \frac{1}{2}z^{-1}\right)X(z).$$

Dividing both sides by $X(z)$ yields

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{2} + \frac{1}{2}z^{-1}.$$

This approach is often the easiest way to find $H(z)$, particularly for IIR filters.

Pole-zero plots

Example. Continuing the previous example, notice that

$$H(z) = \frac{1}{2} + \frac{1}{2}z^{-1} = \frac{1}{2}\frac{z+1}{z}.$$

- The "1/2" is called the **gain** of the filter.
- The roots of the numerator polynomial are called the zeros of the system. In this example the only zero is at $z = -1$.
- The roots of the denominator polynomial are called the zeros of the system. In this example the only pole is at $z = 0$.

We display the poles and zeros graphically using a **pole-zero plot**.
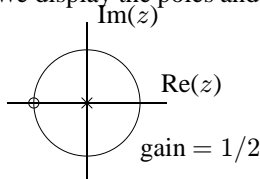
More generally, hereafter we will focus on **rational system functions** which are *ratios of polynomials*

$$H(z) = \frac{B(z)}{A(z)} = g\frac{\prod_i(z - z_i)}{\prod_j(z - p_j)} = g\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}.$$

$B(z)$ is a Mth-order polynomial with M roots $\{z_i, i = 1, \ldots, M\}$ called the **zeros** of $H(z)$, because $H(z_i) = 0$.
$A(z)$ is a Nth-order polynomial with N roots $\{p_j, j = 1, \ldots, N\}$ called the **poles** of $H(z)$, because $H(p_j)$ "=" ∞.

Hereafter, often we will describe filters using the **pole-zero plot** as the graphical representation.

The pole-zero plot (and gain) uniquely describe any rational system function. (New, and final, representation!)

Example. A filter has the following pole-zero plot. Determine $h[n]$.
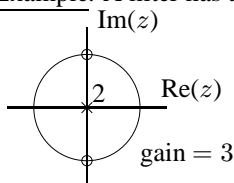
$$H(z) = 3\frac{(z-j)(z+j)}{z^2} = 3\frac{z^2+1}{z^2} = 3 + 3z^{-2}.$$

Thus, by coefficient matching, $h[n] = 3\delta[n] + 3\delta[n-2]\,.$

System functions of causal FIR filters

Why were there poles only at the origin in the preceding two examples? Because FIR filters only have poles at the origin.

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \quad \Longleftrightarrow \quad H(z) = \sum_{k=0}^{M} b_k z^{-k} = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}.$$

Rewriting in two forms:

$$H(z) = \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_{M-1} z + b_M}{z^M} = b_0 \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{z^M},$$

where the z_k's denote the M roots of the numerator polynomial, *i.e.*, the M zeros of the system. Use MATLAB's `roots` function.

Fact. A Mth-order causal FIR filter with $b_0 \neq 0$ and $b_M \neq 0$, has M zeros and has M poles at $z = 0$.

We consider only the usual case of real filters (so the filter coefficients, the b_k's, are real numbers).
Then by the **Fundamental theorem of algebra**, all of the roots of the numerator polynomial are either real, or come in complex conjugate pairs.
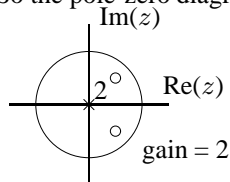
Fact. The zeros (and poles) of a real system are either real or come in complex conjugate pairs.

Example. The system function for a FIR filter with impulse response $h[n] = 2\delta[n] - 2\delta[n-1] + \delta[n-2]$ is:

$$H(z) = 2 - 2z^{-1} + z^{-2} = 2\frac{z^2 - z + \frac{1}{2}}{z^2} = 2\frac{\left[z - \frac{1}{2}(1+\jmath)\right]\left[z - \frac{1}{2}(1-\jmath)\right]}{z^2},$$

because the roots of the numerator polynomial $Az^2 + Bz + C$ are at $z_{\pm} = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2B} = \frac{1}{2}(1 \pm \jmath)$.
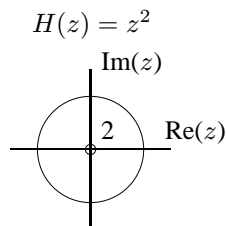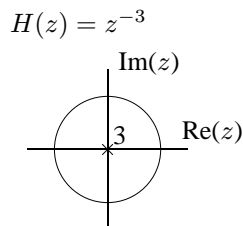
So the pole-zero diagram is:

gain = 2

The zeros are in complex-conjugate pairs.

What about noncausal FIR filters? (Number of poles at the origin varies.)

Examples.

$$H(z) = z^2 \qquad\qquad\qquad H(z) = z^{-3}$$

2 poles at ∞ 3 zeros at ∞

For FIR filters, causal or not, any poles are at $z = 0$.

The pole-zero plot and frequency response

Why is the pole-zero plot useful? System design via frequency response!

Relation between **frequency response** and **system function**: $\boxed{\mathcal{H}(\hat{\omega}) = H(z)\Big|_{z=e^{\jmath\,\hat{\omega}}} = H\!\left(e^{\jmath\,\hat{\omega}}\right).}$

So the **magnitude response**

$$|\mathcal{H}(\hat{\omega})| = \left|H\!\left(e^{\jmath\,\hat{\omega}}\right)\right|$$

From the general expression for $H(z)$ above, the magnitude response of an FIR filter is

$$|\mathcal{H}(\hat{\omega})| = \left|H\!\left(e^{\jmath\,\hat{\omega}}\right)\right| = \left|b_0 \frac{(e^{\jmath\,\hat{\omega}} - z_1)(e^{\jmath\,\hat{\omega}} - z_2)\cdots(e^{\jmath\,\hat{\omega}} - z_M)}{(e^{\jmath\,\hat{\omega}})^M}\right| = |b_0|\left|e^{\jmath\,\hat{\omega}} - z_1\right|\left|e^{\jmath\,\hat{\omega}} - z_2\right|\cdots\left|e^{\jmath\,\hat{\omega}} - z_M\right|.$$

So the magnitude response is simply the product of the distances from the point $e^{\jmath\,\hat{\omega}}$ (on the unit circle) to all the zeros. In particular, whenever $e^{\jmath\,\hat{\omega}}$ is *close* to a zero of the system function, the magnitude response will be reduced. And if $e^{\jmath\,\hat{\omega}} = z_m$ for one of the M zeros, then the magnitude response will be exactly zero.
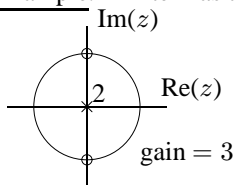
This makes it fairly easy to design filters that remove certain frequency components completely.

And, given a pole-zero plot, we can roughly sketch the magnitude response!
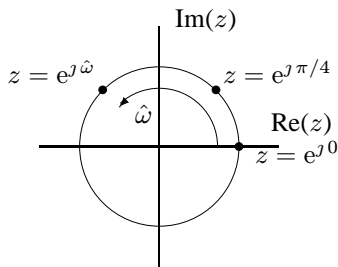
The unit circle in the z-domain corresponds to the interval $-\pi \leq \hat{\omega} \leq \pi$ for the frequency domain.
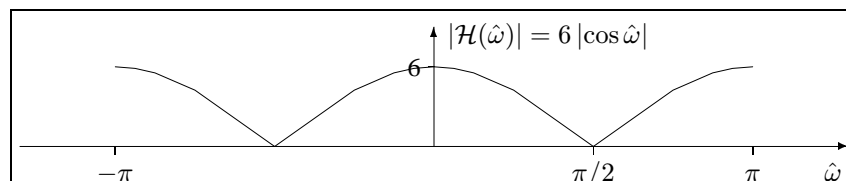
Example. A filter has the following pole-zero plot. Sketch its magnitude response.

For $\hat{\omega} = 0$, the distance to each zero is $\sqrt{2}$, so the magnitude response is $3\sqrt{2}\sqrt{2} = 6$.
Likewise for $\hat{\omega} = \pm\pi$.
As $\hat{\omega}$ increases from 0 towards $\pi/2$, the distance to the upper zero decreases and the magnitude response drops towards zero.

Analytically: $H(z) = 3 + 3z^{-2}$ so $\mathcal{H}(\hat{\omega}) = 3 + 3e^{-\jmath\,2\hat{\omega}} = e^{-\jmath\,\hat{\omega}}6\cos\hat{\omega}$, so the magnitude response is $|\mathcal{H}(\hat{\omega})| = 6\left|\cos\hat{\omega}\right|$.
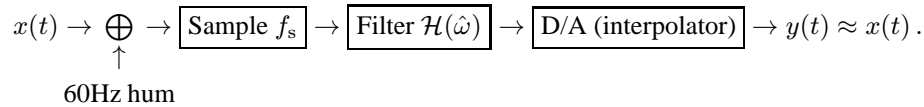For what sampling rate f_s would this filter remove 60Hz hum? $f_\mathrm{s} = 240$Hz

The book also discusses how to analyze the phase response graphically.
This topic will be considered in more detail in EECS 451.

First FIR filter design attempt: 60Hz notch filter.

Equipped with the concepts developed thus far, we can finally attempt our first **filter design**!

In many applications, such as measuring brain signals, "60Hz hum" from AC power lines contaminates the signal $x(t)$. Another example is home computer networks that operate over house AC wiring. To reduce the resulting interference, often we need to pass the signal through a filter that removes the 60Hz component while leaving other frequency components relatively unaffected. Such filtering can be done using RLC circuits, or by applying digital signal processing.

$$x(t) \to \bigoplus \to \boxed{\text{Sample } f_\text{s}} \to \boxed{\text{Filter } \mathcal{H}(\hat{\omega})} \to \boxed{\text{D/A (interpolator)}} \to y(t) \approx x(t).$$
$$\uparrow$$
$$\text{60Hz hum}$$

For simplicity of the algebra, we suppose that the sampling rate is $f_\text{s} = 480\text{Hz}$.
What is the corresponding digital frequency? $\hat{\omega} = 2\pi f / f_\text{s} = 2\pi 60/480 = \pi/4$.

What is the ideal frequency response $\mathcal{H}(\hat{\omega})$?
(Picture) of ideal $\mathcal{H}(\hat{\omega})$ with nulls at $\pm\pi/4$.
This is called a **notch filter**.

Design 1

Where do we put zeros in the pole-zero diagram to eliminate the frequency component $\hat{\omega} = \pm\pi/4$?
See pole-zero diagram below with two zeros at $\pm e^{\jmath\pi/4}$.

Where do the poles go? Two at origin needed for causal FIR filter.

This system design looks promising, since it will certainly eliminate the 60Hz ($\hat{\omega} = \pm\pi/4$) frequency component completely.
Next we find the **impulse response** (needed for implementation) and the **frequency response** (to see if we met our design goal).

Choosing gain=1 for now, we see from the pole-zero plot:

$$H(z) = \frac{\left[z - e^{\jmath\pi/4}\right]\left[z - e^{-\jmath\pi/4}\right]}{z^2} = \frac{z^2 - z2\cos(\pi/4) + 1}{z^2} = 1 - z^{-1}2\cos(\pi/4) + z^{-2} = 1 - z^{-1}\sqrt{2} + z^{-2}.$$
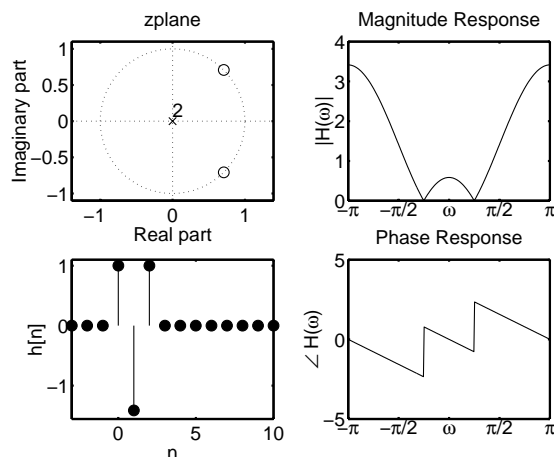
Thus the impulse response is: $h[n] = \delta[n] - \sqrt{2}\delta[n-1] + \delta[n-2]$. This is a very simple filter! It can be implemented using the following simple diffeq:

$$y[n] = x[n] - \sqrt{2}x[n-1] + x[n-2].$$

The frequency response of this FIR filter is

$$\mathcal{H}(\hat{\omega}) = 1 - \sqrt{2}e^{-\jmath\hat{\omega}} + e^{-\jmath2\hat{\omega}} = e^{-\jmath\hat{\omega}}(2\cos\hat{\omega} - \sqrt{2}).$$

Here is a plot of the **magnitude response** and **phase response** of this system, computed using MATLAB's `zplane` and `freqz` functions.

Design 2

Our first design does remove the $\hat{\omega} = \pm\pi/4$ frequency component completely, but unfortunately it also has the side effect of amplifying the high frequencies relative to the low frequencies.

How can we attenuate the high frequencies somewhat?

From the general expression for $H(z)$ above, the magnitude response is

$$|\mathcal{H}(\hat{\omega})| = \left|H\left(e^{\jmath\hat{\omega}}\right)\right| = \left|\frac{(e^{\jmath\hat{\omega}} - z_1)(e^{\jmath\hat{\omega}} - z_2)\cdots(e^{\jmath\hat{\omega}} - z_M)}{(e^{\jmath\hat{\omega}})^M}\right| = |e^{\jmath\hat{\omega}} - z_1||e^{\jmath\hat{\omega}} - z_2|\cdots|e^{\jmath\hat{\omega}} - z_M|.$$

So the magnitude response is simply the product of the distances from the point $e^{\jmath\hat{\omega}}$ (on the unit circle) to all the zeros. In particular, whenever $e^{\jmath\hat{\omega}}$ is *close* to a zero of the system function, the magnitude response will be reduced.

So to reduce (but not eliminate) high frequency components ($\hat{\omega} \approx \pm\pi$), we can place a zero somewhere near $z = e^{\jmath\pi} = -1$.

Comparing $\mathcal{H}(0) = H(1)$ to $\mathcal{H}(\pi) = H(-1)$, one can show that a zero at $z = -\sqrt{2}$ will equalize the gain at $\hat{\omega} = 0$ and $\hat{\omega} = \pi$.

Here is the resulting frequency response.
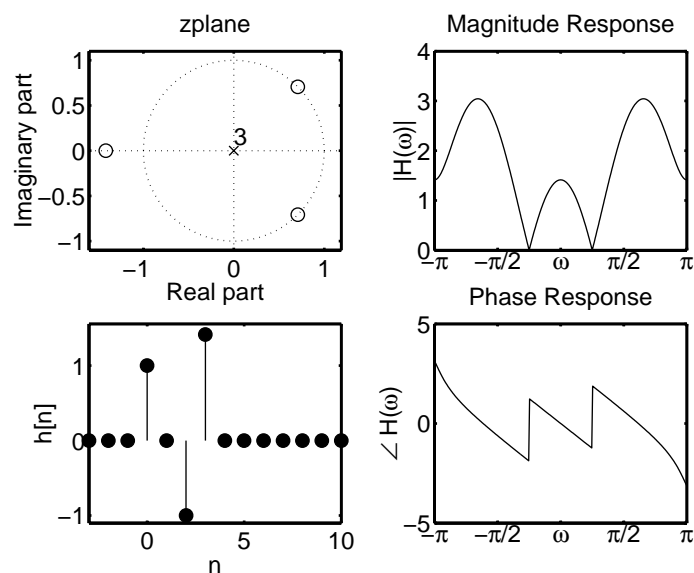
Well, we have equalized the gain at $\hat{\omega} = 0$ and $\hat{\omega} = \pi$, but we are still far from our ideal frequency response!

What is the output of this filter if the input signal is $x[n] = 3 + \cos(\frac{\pi}{4}n) + 7(-1)^n$?

Recall that

$$x[n] = \cos(\hat{\omega}n + \phi) \to \boxed{\mathcal{H}(\hat{\omega})} \to y[n] = |\mathcal{H}(\hat{\omega})|\cos(\hat{\omega}n + \phi + \angle\mathcal{H}(\hat{\omega})).$$

For the system above,

$$\begin{aligned}
H(z) &= \frac{\left[z - e^{\jmath\pi/4}\right]\left[z - e^{-\jmath\pi/4}\right]\left[z + \sqrt{2}\right]}{z^3} = [1 - z^{-1}\sqrt{2} + z^{-2}][1 + \sqrt{2}z^{-1}] = 1 - z^{-2} + \sqrt{2}z^{-3} \\
\mathcal{H}(\hat{\omega}) &= H\left(e^{\jmath\hat{\omega}}\right) = [1 - \sqrt{2}e^{-\jmath\hat{\omega}} + e^{-\jmath 2\hat{\omega}}][1 + \sqrt{2}e^{-\jmath\hat{\omega}}].
\end{aligned}$$

The frequencies of the input signal $x[n]$ above are $\hat{\omega} = 0$, $\hat{\omega} = \pi/4$, and $\hat{\omega} = \pi$.

Substituting into $\mathcal{H}(\hat{\omega})$ we have $\mathcal{H}(0) = (2 - \sqrt{2})(1 + \sqrt{2}) = \sqrt{2}$, $\mathcal{H}(\pi/4) = 0$ (due to zeros at $\pm\pi/4$ on unit circle), $\mathcal{H}(\pi) = (2 + \sqrt{2})(1 - \sqrt{2}) = -\sqrt{2} = \sqrt{2}e^{\jmath\pi}$.

Thus the output signal is

$$y[n] = 3\sqrt{2} + 0 + \sqrt{2}\cos(\pi n + \pi).$$

A magnitude response plot shows the response to an *eternal* sinusoid.
What about "suddenly turned on" signals $\cos(\hat{\omega}n)u[n]$, (*e.g.*, after system first started)?
After short initial **transient response**, the output signal approaches the **steady state** response.

Design 3

How well can we do with a more sophisticated FIR filter?
Here is a $M = 28$ filter designed by MATLAB's remez function.
The frequency response plots were made using MATLAB's freqz function.
The impulse response plots were made using MATLAB's filter function.
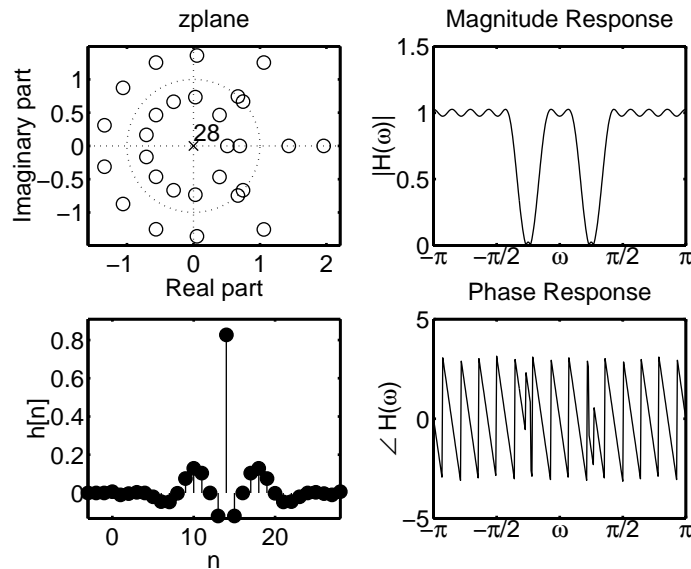
This is better since it has reasonably uniform gain in the passbands, but the "notch" may be somewhat wide.

Design 4

How can we get even closer to the ideal frequency response?
One way is to use the filter with the following pole-zero plot and frequency response.
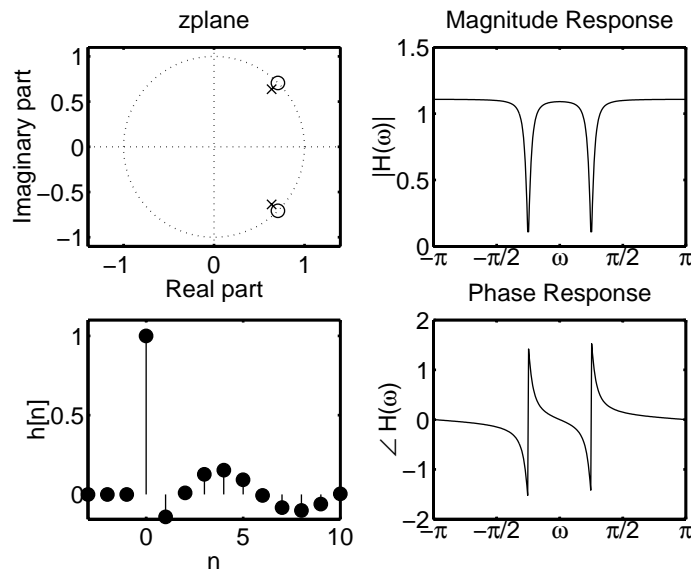
What is fundamentally different about this filter?
It is an **IIR filter**, since there are poles not at $z = 0$.

Even though we have not yet studied IIR filters in detail, we can use what we have learned so far to find
- the **system function** $H(z)$,
- the **diffeq**,
- and the **frequency response**.

However, we do not yet know how to find the **impulse response** of an IIR filter. That comes in the next chapter.

Assuming that the gain=1, from the pole-zero we have that the **system function** is:

$$H(z) = \frac{[z - e^{\jmath\pi/4}][z - e^{-\jmath\pi/4}]}{[z - re^{\jmath\pi/4}][z - re^{-\jmath\pi/4}]} = \frac{z^2 - z2\cos(\pi/4) + 1}{z^2 - z2r\cos(\pi/4) + r^2} = \frac{1 - z^{-1}2\cos(\pi/4) + z^{-2}}{1 - z^{-1}2r\cos(\pi/4) + r^2z^{-2}} = \frac{Y(z)}{X(z)},$$

where r denotes the distance of the pole from the origin.

To find the **diffeq**, we first **cross multiply** as follows:

$$Y(z)\left[1 - z^{-1}2r\cos(\pi/4) + r^2z^{-2}\right] = X(z)\left[1 - z^{-1}2\cos(\pi/4) + z^{-2}\right].$$

Now (using linearity and shift property of z-transform), we convert back to the time domain:

$$y[n] - 2r\cos(\pi/4)y[n-1] + r^2y[n-2] = x[n] - 2\cos(\pi/4)x[n-1] + x[n-2].$$

Rearranging yields the following **recursive** expression:

$$y[n] = 2r\cos(\pi/4)y[n-1] - r^2y[n-2] + x[n] - 2\cos(\pi/4)x[n-1] + x[n-2].$$

The output at time n depends on the two past output values *and* on the current input and 2 past input values. This is very easy to implement! Much easier than a 28-tap FIR filter.

To understand the **frequency response** qualitatively, we return to the first **factored form** of the system function above:

$$\mathcal{H}(\hat{\omega}) = H\left(e^{\jmath\hat{\omega}}\right) = \frac{[e^{\jmath\hat{\omega}} - e^{\jmath\pi/4}][e^{\jmath\hat{\omega}} - e^{-\jmath\pi/4}]}{[e^{\jmath\hat{\omega}} - re^{\jmath\pi/4}][e^{\jmath\hat{\omega}} - re^{-\jmath\pi/4}]}.$$

In particular, we find the **magnitude response** by taking the magnitude of both sides:

$$|\mathcal{H}(\hat{\omega})| = \frac{|e^{\jmath\hat{\omega}} - e^{\jmath\pi/4}||e^{\jmath\hat{\omega}} - e^{-\jmath\pi/4}|}{|e^{\jmath\hat{\omega}} - re^{\jmath\pi/4}||e^{\jmath\hat{\omega}} - re^{-\jmath\pi/4}|}.$$

In words, this means that for any frequency $\hat{\omega}$, the magnitude response is the product of all the distances from the point $e^{\jmath\hat{\omega}}$ on the unit circle to all the zeros, divided by the product of all the distances from the point $e^{\jmath\hat{\omega}}$ to all the poles.
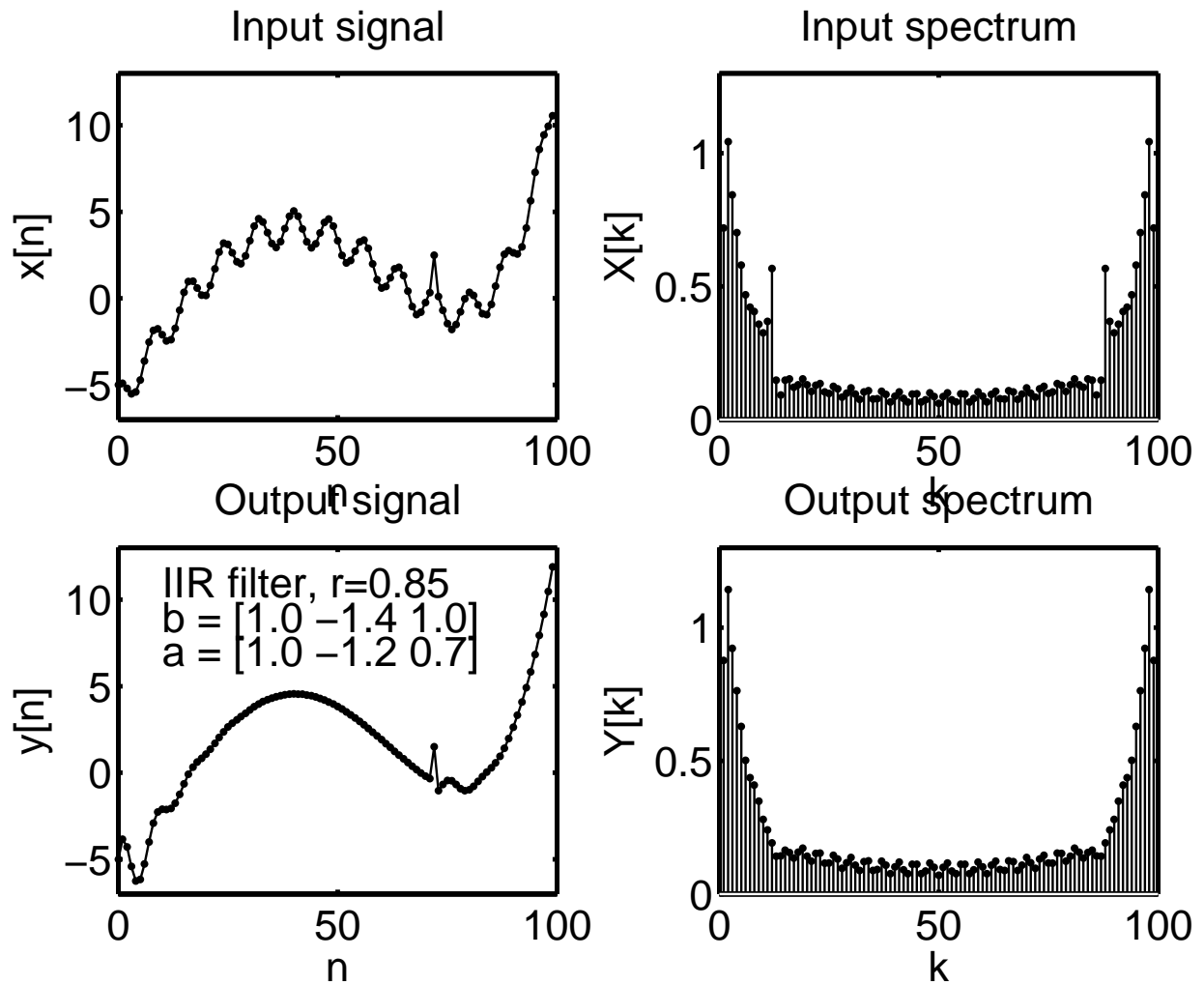Being close to a zero decreases the magnitude response; being close to pole increases the magnitude response.

For any frequency $\hat{\omega}$ that is far from $e^{\jmath\pi/4}$ in the notch filter example, the distance from $e^{\jmath\hat{\omega}}$ to the zero and to its neighboring pole is almost the same, so the ratio is nearly unity and thus the magnitude response is nearly unity. The zero is slightly further away, so the magnitude response is slightly more than unity.

How could we make the response closer to unity far from the zero?
Move the pole closer to the zero. (But, as we will see in the next chapter, this makes the transient response "longer," *i.e.*, the impulse response decays to zero slower, which can be undesirable in some applications.

The following figure shows the result of applying this filter to the example data considered much earlier. The **transient response** of the IIR filter is clearly visible.

Input signal

Input spectrum

Output signal

IIR filter, r=0.85
b = [1.0 −1.4 1.0]
a = [1.0 −1.2 0.7]

Output spectrum

Interconnection of LTI systems

Engineers build complicated systems by connecting simpler components.

There are three principal ways of connecting two LTI systems: series, parallel, and in a feedback loop.

Series connection

Here are two LTI systems connected in **series** or in **cascade**:

$$x[n] \rightarrow \boxed{H_1(z)} \overset{y_1[n]}{\longrightarrow} \boxed{H_2(z)} \rightarrow y[n] \,.$$

In the time domain, we have seen that that overall system is LTI with impulse response $h[n] = h_1[n] * h_2[n]$.

Since time-domain convolution corresponds to multiplication in the z-domain, the overall system function is

$$H(z) = H_1(z) \cdot H_2(z) \,,$$

the *product* of the two individual system functions.

Example. Application: channel equalization for wireless communications. $H_1(z)$ represents the channel (attenuation, multipath) and $H_2(z)$ represents the equalizer that is designed to "undo" (invert) the distortions induced by the channel. Ideally, we would have $H_2 = 1/H_1$. We must estimate the channel response H_1. We sent a "training sequence" with known $x[n]$, observe the output $y_1[n]$, and then use $H_1(z) = Y_1(z)/X(z)$. These ideas are used in digital communications.

Example. Application: (greatly simplified) Dolby noise reduction...
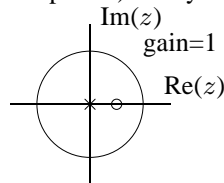
High-boost for record: High-reduce for playback (**inverse system**: IIR!):
(Picture) of frequency responses.
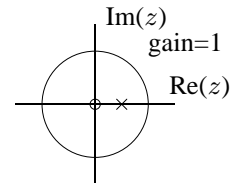
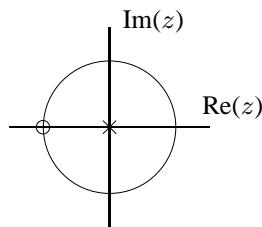Example. Putting two unit-sample delays in series is equivalent to a two-sample delay:

$$x[n] \rightarrow \boxed{z^{-1}} \rightarrow \boxed{z^{-1}} \rightarrow y[n] \text{ is equivalent to } x[n] \rightarrow \boxed{z^{-2}} \rightarrow y[n] \,.$$

Example. $H_1(z) = g_1(1 + z^{-1}) = g_1 \frac{z+1}{z}$ and $H_2(z) = g_2(1 - z^{-1}) = g_2 \frac{z-1}{z}$.
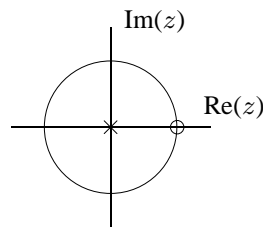In series:

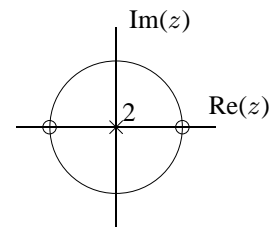$$H(z) = H_1(z)\, H_2(z) = g_1 g_2 \frac{z+1}{z} \frac{z-1}{z} = g_1 g_2 \frac{(z+1)(z-1)}{z^2} \,.$$
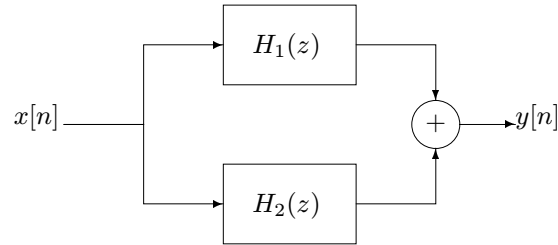
Pole-zero plots:

$H_1(z):$ $H_2(z):$ $H:$

For **rational** system functions:

$$H(z) = H_1(z)\, H_2(z) = \frac{B_1(z)}{A_1(z)} \frac{B_2(z)}{A_2(z)} \,.$$

So the zeros of $H(z)$ are all the zeros of the two systems connected in zeros, and likewise for the poles, except in cases where $B_1(z)$ and $A_2(z)$ share a common root, or where $B_2(z)$ and $A_1(z)$ share a common root, which is called **pole-zero cancellation**.

Parallel connection

$Y(z) = H_1(z) X(z) + H_2(z) X(z) = (H_1(z) + H_2(z))X(z)$, so the overall system function is

$$H(z) = H_1(z) + H_2(z) = \frac{B_1(z)}{A_1(z)} + \frac{B_2(z)}{A_2(z)} = \frac{B_1(z) A_2(z) + B_2(z) A_1(z)}{A_1(z) A_2(z)}.$$

In this case we see that the parallel connected system has all the poles of the two component systems (unless there is pole-zero cancellation.

But we can say very little if anything about the zeros in general.
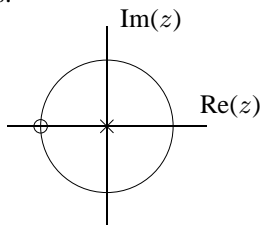
<u>Example.</u> $H_1(z) = g_1(1 + z^{-1}) = g_1 \frac{z+1}{z}$ and $H_2(z) = g_2(1 - z^{-1}) = g_2 \frac{z-1}{z}$.
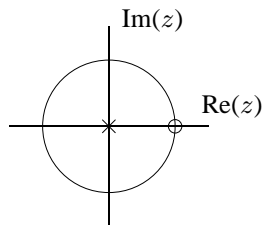In parallel:

$$H(z) = H_1(z) + H_2(z) = g_1 \frac{z+1}{z} + g_2 \frac{z-1}{z} = \frac{(g_1 + g_2)z + (g_2 - g_1)}{z}.$$

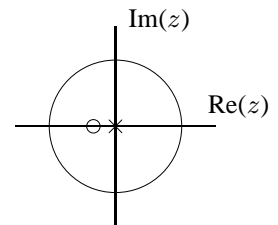Single pole at $z = 0$ and zero at $z = (g_1 - g_2)/(g_1 + g_2)$.

Pole-zero plots:

$H_1(z)$: $\qquad\qquad$ $H_2(z)$: $\qquad\qquad$ H (for $g_1 = 1$, $g_2 = 2$):

Feedback connection

<u>Example.</u> automobile cruise control.

(Picture of system $H_1(z)$ with negative feedback $H_f(z)$)

Derivation: $Y = H_1(X - H_f Y)$ so $(1 + H_f)Y = H_1 X$ so

$$H(z) = \frac{H_1(z)}{1 + H_1(z) H_f(z)}.$$

Location of poles and zeros of $H(z)$ is complicated; indeed, part of the purpose of feedback is to stabilize and otherwise unstable system (*e.g.*, Harriet airplane). Even if $H_1(z)$ has poles that are outside the unit circle, which means it is an unstable system, it can still be the case that $H(z)$ has all its poles inside the unit circle, meaning that it is stable, if the feedback system $H_f(z)$ is designed properly.

<u>Example.</u> $H_1(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$. If $a > 1$, then the system is unstable.
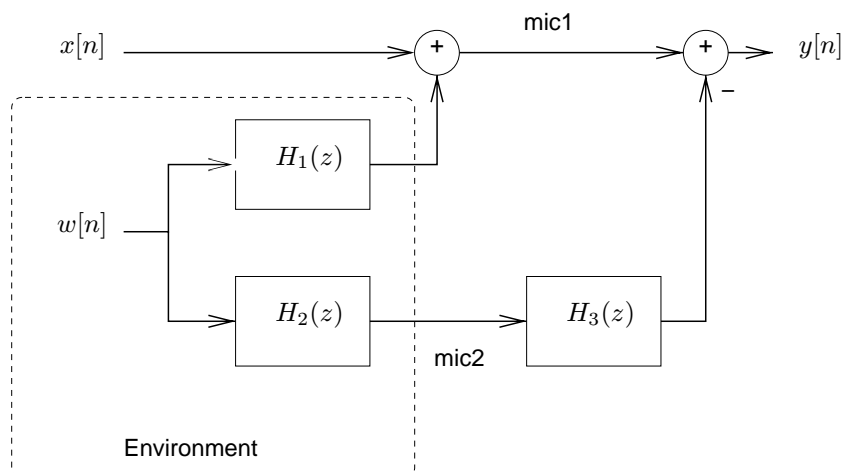
The simplest possible feedback controller would be $H_f(z) = g$ for some constant g.

$$H(z) = \frac{H_1(z)}{1 + H_1(z) H_f(z)} = \frac{\frac{z}{z-a}}{1 + \frac{z}{z-a}g} = \frac{z}{z - a + zg} = \frac{z}{z(1+g) - a},$$

so the system with feedback has a pole at $z = a/(1 + g)$.
If $g > a - 1$, then the pole will be "moved" to within the unit circle, thereby stabilizing the system.

Example. Noise cancellation. (Explained in lecture)

For perfect noise cancellation, we need $H_3(z) = -H_1(z)/H_2(z)$.
If $H_1(z)$ is FIR, then $H_3(z)$ will be IIR, which again brings us to the next chapter...

Summary and other topics

system function $H(z)$: z-transform of impulse response.

pole-zero plot / filter design.

Response to suddenly applied complex exponential signals and sinusoids?

Design of FIR **bandpass filter**? (zeros around unit circle except in passband)

(Picture) of ideal $\mathcal{H}(\hat{\omega})$

(Picture) pole-zero plot