Instructions:

- Answer on this questionnaire
- Print your name
- Sign the pledge below
- Closed book and notes
- One 8 1/2 x 11 sheet of paper allowed
- Calculators allowed
- Read the questions carefully.
- Problems 1 to 10 are multiple-choice. Each has 7 points. No partial credit will be given.
- In Problems 11 and 12, partial credit will be given. You must show your derivations/calculations to get full credit.

Name:

PLEDGE: I have neither given nor received any aid on this exam, nor have I concealed any violations of the Honor Code.

SIGNATURE:

DO NOT TURN THIS PAGE OVER UNTIL TOLD TO DO SO!
Good Luck!

Mark your answers for Problems (1)–(10)

(1)	(a)	(b)	(c)	(d)	(e)
(2)	(a)	(b)	(c)	(d)	(e)
(3)	(a)	(b)	(c)	(d)	(e)
(4)	(a)	(b)	(c)	(d)	(e)
(5)	(a)	(b)	(c)	(d)	(e)
(6)	(a)	(b)	(c)	(d)	(e)
(7)	(a)	(b)	(c)	(d)	(e)
(8)	(a)	(b)	(c)	(d)	(e)
(9)	(a)	(b)	(c)	(d)	(e)
(10)	(a)	(b)	(c)	(d)	(e)
Subtotal					
(11)					
(12)					
Total					

(1) The fundamental period of the following discrete-time signal

$$2\cos(\frac{\pi}{4}n + 0.1) + \cos(\frac{3\pi}{7}n + 0.3)$$

is

(a) 8
(b) 14
(c) 28
(d) ∞
(e) None of the above

Answer: (e)

The signal consists of two sinusoids. The component signals are periodic: Let N_1 and N_2 denote their respective periods (they ust be positive integers).

$$\frac{\pi}{4}N_1 = 2\pi m,$$
$$\frac{3\pi}{7}N_2 = 2\pi l,$$

where m and l are positive integers. This implies that

$$N_1 = 8m, \quad N_2 = 14l/3.$$

Since N_1 and N_2 are positive integers,

$$N_1 = 8, 16, \ldots,$$
$$N_2 = 14, 28, \ldots.$$

Therefore, common multiples of 8 and 14 will be periods of the given signal:

$$56, 112, \ldots.$$

(2) A real discrete-time signal $x[n]$ is periodic with the fundamental period 8 and some of its 8-point DFT coefficients are given below.

$$X_0 = 0, \quad X_1 = 1, \quad X_2 = 1 - j, \quad X_3 = 1 + j2.$$

Then X_6 is

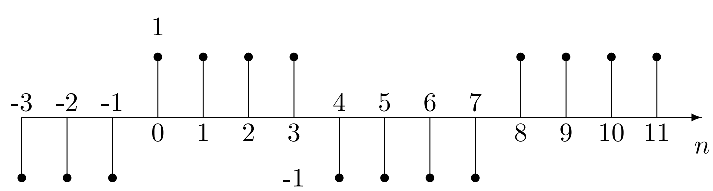
(a) 0
(b) 1
(c) $1 + j$
(d) $1 - j2$
(e) Not enough information is given

Answer: (c)

A real signal has the complex conjugate property: $X_{N-k} = X_k^*$. Then

$$X_6 = X_{8-2} = X_2^* = 1 + j.$$

(3) A real discrete-time signal $x[n]$ is periodic with the fundamental period 8.



Let X_k be the DFT coefficients of the 8-point DFT of $x[n]$. Then the value of $\sum_{k=1}^{7} \left|X_k\right|^2$ equals

(a) $\frac{1}{\sqrt{2}}$
(b) 1
(c) $\sqrt{2}$
(d) $2\sqrt{2}$
(e) None of the above

Answer: (b)

Parseval's theorem states that

$$\frac{1}{8}\sum_{n=0}^{7} \left|x[n]\right|^2 = \sum_{k=0}^{7} \left|X_k\right|^2.$$

Note that for the given signal

$$X_0 = \frac{1}{8}\sum_{k=0}^{7} x[n] = 0.$$

Therefore,

$$\frac{1}{8}\sum_{n=0}^{7} \left|x[n]\right|^2 = \sum_{k=0}^{7} \left|X_k\right|^2 = \underbrace{\left|X_0\right|^2}_{0} + \sum_{k=1}^{7} \left|X_k\right|^2.$$

By evaluation, we get

$$\frac{1}{8}\sum_{n=0}^{7} \left|x[n]\right|^2 = 1.$$

(4) The signal

$$10\cos\left(2\pi 25t + \frac{\pi}{4}\right),$$

when sampled at $f_s = 20$ samples/sec, will alias to

(a) $10\cos\left(2\pi 5t - \frac{\pi}{4}\right)$
(b) $10\cos\left(2\pi 5t + \frac{\pi}{4}\right)$
(c) $10\cos\left(2\pi 15t - \frac{\pi}{4}\right)$
(d) $10\cos\left(2\pi 15t + \frac{\pi}{4}\right)$
(e) None of the above

Answer: (b)

Aliases of $10\cos\left(2\pi 25t + \frac{\pi}{4}\right)$

$$10\cos\left(2\pi(25 + f_s l)t + \frac{\pi}{4}\right) \quad \text{or} \quad 10\cos\left(2\pi(f_s m - 25)t - \frac{\pi}{4}\right),$$

where l and m are integers. Among these candidates the answer is the one with frequency less than half the sampling frequency, $f_s/2 = 10$. Therefore, from the following candidates,

$$\begin{aligned} l = -1, 0, 1, \ldots &\implies f = 5, 25, 45, \ldots, \\ m = 2, 3, 4, \ldots &\implies f = 15, 35, 55, \ldots. \end{aligned}$$

the frequncy that is less than $f_s/2 = 10$ is obtained when $l = -1$ and the corresponding signal is $10\cos\left(2\pi 5t + \frac{\pi}{4}\right)$.

(5) The ideal interpolation of

$$\cos(\frac{\pi}{4}n + 0.1) + \cos(\frac{7\pi}{4}n - 0.1)$$

using the sampling period $T_s = 1/f_s = 0.01$ will produce

(a) $\cos(\frac{\pi}{4}100t + 0.1) + \cos(\frac{7\pi}{4}100t - 0.1)$
(b) $\cos(\frac{\pi}{4}100t + 0.1) - \cos(\frac{7\pi}{4}100t + 0.1)$
(c) $2\cos(\frac{\pi}{4}100t + 0.1)$
(d) $2\cos(\frac{\pi}{4}100t - 0.1)$
(e) None of the above

Answer: (c)

We note that

$$\cos(\hat{\omega}n + \phi) \implies \cos(\hat{\omega}f_s t + \phi)$$

if $0 \le \hat{\omega} < \pi$. Since the second term of the given signal

$$\cos(\frac{7\pi}{4}n - 0.1) = \cos((\frac{7\pi}{4} - 2\pi)n - 0.1) = \cos(-\frac{\pi}{4}n - 0.1) = \cos(\frac{\pi}{4}n + 0.1),$$

the given signal is, in effect,

$$\cos(\frac{\pi}{4}n + 0.1) + \cos(\frac{7\pi}{4}n - 0.1) = 2\cos(\frac{\pi}{4}n + 0.1).$$

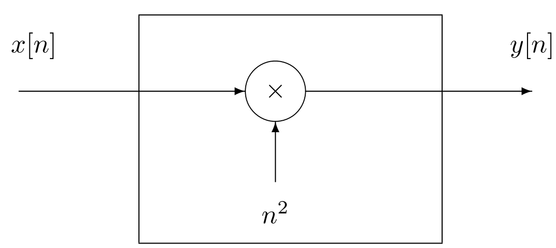
Therefore, the ideal reconstruction will be

$$2\cos(\frac{\pi}{4}n + 0.1) \implies 2\cos(\frac{\pi}{4}100t + 0.1).$$

(6) The system whose input/output relationship is given by

$$y[n] = n^2 x[n]$$

is



(a) linear and time-invariant
(b) nonlinear and time-invariant
(c) linear and time-varying
(d) noninear and time-varying
(e) Not enough information is given

Answer: (c)

The system is linear and time-varying.

(i) Linearity

$$\begin{aligned}\mathcal{T}\{\alpha x_1[n] + \beta x_2[n]\} &= n^2\big(\alpha x_1[n] + \beta x_2[n]\big)\\ &= \alpha n2x_1[n] + \beta n^2 x_2[n]\\ &= \alpha\mathcal{T}\{x_1[n]\} + \beta\mathcal{T}\{x_2[n]\}.\end{aligned}$$

(ii) Time-varying

$$\begin{aligned}\mathcal{T}\{x[n-n_0]\} &= n^2 x[n-n_0]\\ &\neq (n-n_0)^2 x[n-n_0] = y[n-n_0].\end{aligned}$$

(7) The condition for the system

$$y[n] = Ax^2[n] + Bx[n-3] + C$$

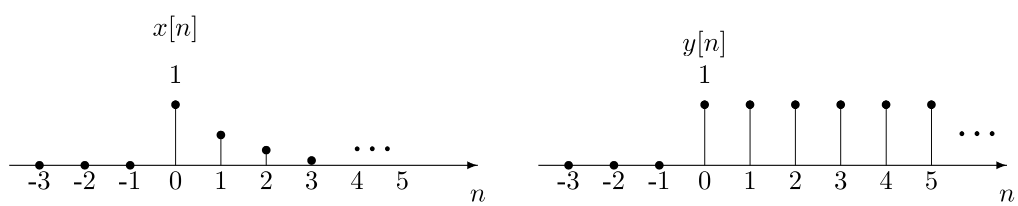
to be linear and time-invariant is
(a) $ABC \neq 0$.
(b) $A = B = 0$ and $C \neq 0$.
(c) $A \neq 0$ and $B = C = 0$.
(d) $C = 0$.
(e) $A = C = 0$.

Answer: (e)

A must be zero, because otherwise the system will create a cross term, making it nonliner. Also C must be zero, because a nonzero constant term violate linearity.

(8) The convolution of the following two signals

$$x[n] = \left(\frac{1}{2}\right)^n u[n] \quad \text{and} \quad y[n] = u[n]$$



is given by, for $n \geq 0$,
(a) $2 - (\frac{1}{2})^n$
(b) $2 + (\frac{1}{2})^n$
(c) $2 - (\frac{1}{2})^{n+1}$
(d) $2 + (\frac{1}{2})^{n+1}$
(e) None of the above

Answer: (a)

$$
\begin{aligned}
x[n] * y[n] &= \sum_{k=-\infty}^{\infty} x[k]y[n-k] \\
&= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k \underbrace{u[k]}_{=1, k\geq 0} \underbrace{u[n-k]}_{=1, n \geq k} \\
&= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \quad n \geq 0 \\
&= \frac{1-(1/2)^{n+1}}{1-1/2} = 2 - \left(\frac{1}{2}\right)^n.
\end{aligned}
$$

(9) An LTI filter is described by the following difference equation:

$$y[n] = x[n+1] + 2x[n] + x[n-1].$$

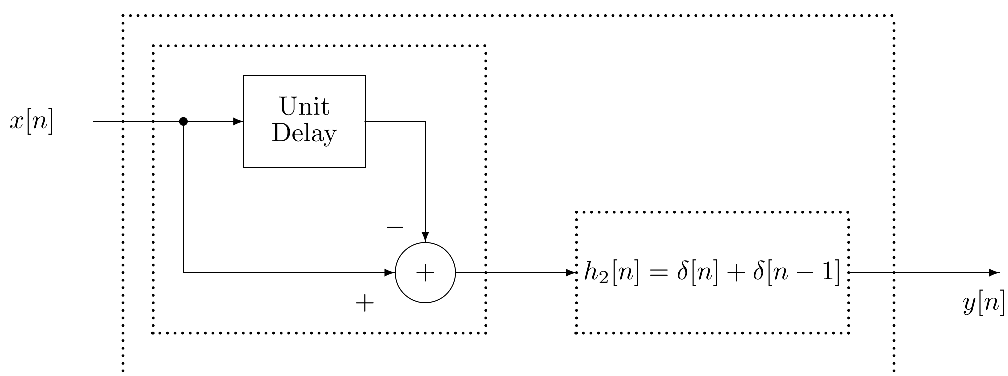
Which of the following statements is *incorrect*?

(a) Its order is 3.
(b) It is noncausal.
(c) Its impulse response $h[n]$ is given by $h[n] = \delta[n+1] + 2\delta[n] + \delta[n-1]$.
(d) $y[n]$ is periodic whenever $x[n]$ is periodic.
(e) $y[n]$ is sinusoidal whenever $x[n]$ is sinusoidal.

Answer: (a)

The order is 2.

(10) The overall impulse response $h[n]$ of the following two cascaded LTI filters



is

(a) $\delta[n] - \delta[n-1]$.
(b) $\delta[n] + \delta[n-1]$.
(c) $\delta[n] - \delta[n-2]$.
(d) $(\delta[n])^2 - (\delta[n-1])^2$.
(e) None of the above.

Answer: (c) (Also (d) is acceptable.)

Two LI filters are cascaded. So,

$$\begin{aligned} h[n] = h_1[n] * h_2[n] &= (\delta[n] - \delta[n-1]) * (\delta[n] + \delta[n-1]) \\ &= \delta[n] - \delta[n-2]. \end{aligned}$$

Choice (d) happens to give the same answer: in general

$$\delta[n-k] = (\delta[n-k])^m, \quad \text{for } m = 1, 2, \ldots.$$

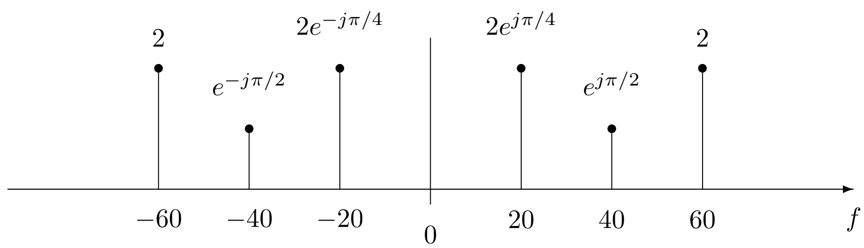
(11) (15 points) The following continuous-time signal is to be sampled and reconstructed.

$$x(t) = 4\cos\left(2\pi 20t + \frac{\pi}{4}\right) + 2\cos\left(2\pi 40t + \frac{\pi}{2}\right) + 4\cos\left(2\pi 60t\right).$$

Let the sampling frequency $f_s = 100$ samples/sec and $x[n] = x(nT_s) = x(n/f_s)$.

(a) Plot the double-sided spectrum of $x(t)$ as a functon of f Hz.

$$\begin{aligned} x(t) &= 4\cos\left(2\pi 20t + \frac{\pi}{4}\right) + 2\cos\left(2\pi 40t + \frac{\pi}{2}\right) + 4\cos\left(2\pi 60t\right) \\ &= 2e^{j\frac{\pi}{4}}e^{j2\pi 20t} + 2e^{-j\frac{\pi}{4}}e^{-j2\pi 20t} + e^{j\frac{\pi}{2}}e^{j2\pi 40t} + e^{-j\frac{\pi}{2}}e^{-j2\pi 40t} + 2e^{j2\pi 60t} + 2e^{-j2\pi 60t}. \end{aligned}$$



(b) Determine whether $x(t)$ can be perfectly reconstructed from $x[n]$ using the ideal interpolation. Justify your answer.
No, because aliasing occurs due to the fact that $f_s = 100 < 2f_{\max} = 2 \cdot 60 = 120$.

(c) Find and simplify the reconstruction $\tilde{x}(t)$ from $x[n]$ using the ideal interpolation.
When sampled at $f_s = 100$ ($T_s = 0.01$), we get

$$\begin{aligned} x[n] &= 4\cos\left(2\pi 20 \cdot 0.01n + \frac{\pi}{4}\right) + 2\cos\left(2\pi 40 \cdot 0.01n + \frac{\pi}{2}\right) + 4\cos\left(2\pi 60 \cdot 0.01n\right) \\ &= 4\cos\left(0.4\pi n + \frac{\pi}{4}\right) + 2\cos\left(0.8\pi n + \frac{\pi}{2}\right) + \underbrace{4\cos\left(1.2\pi n\right)}_{=4\cos\left(-0.8\pi n\right)} \\ &= 4\cos\left(0.4\pi n + \frac{\pi}{4}\right) + \underbrace{2\cos\left(0.8\pi n + \frac{\pi}{2}\right) + 4\cos\left(0.8\pi n\right)}_{\text{combine these}} \\ &= 4\cos\left(0.4\pi n + \frac{\pi}{4}\right) + 4.4721\cos\left(0.8\pi n + 0.4636\right). \end{aligned}$$

Note that each of frequencies in the last expression is less than π. Then the ideal interpolation will reconstruct

$$\begin{aligned} \tilde{x}(t) &= 4\cos\left(0.4\pi 100t + \frac{\pi}{4}\right) + 4.4721\cos\left(0.8\pi 100t + 0.4636\right) \\ &= 4\cos\left(2\pi 20t + \frac{\pi}{4}\right) + 4.4721\cos\left(2\pi 40t + 0.4636\right). \end{aligned}$$

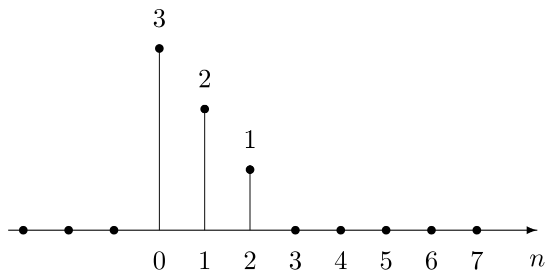
This reconstructed waveform differs from $x(t)$ due to undersampling and hence aliasing.

(12) (20 points) Answer for the LTI filter whose input/output relationship is given by

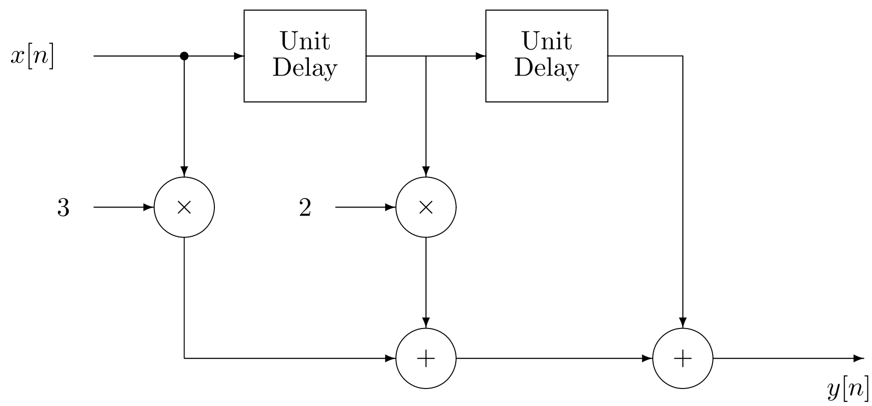
$$y[n] = 3x[n] + 2x[n-1] + x[n-2].$$

(a) Find and plot the impulse response $h[n]$.

$$h[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2].$$



(b) Draw the block diagram implementation in the direct form.



(c) Find the response of the filter to input $x[n] = e^{j(\frac{\pi}{4}n - 0.1)}$ and simplify it.

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{-j0.1}e^{j\frac{\pi}{4}(n-k)} \\
&= e^{-j0.1}e^{j\frac{\pi}{4}n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\frac{\pi}{4}k} \\
&= e^{-j0.1}e^{j\frac{\pi}{4}n}\left(3 + 2e^{-j\frac{\pi}{4}} + e^{-j\frac{\pi}{2}}\right) \\
&= e^{-j0.1}e^{j\frac{\pi}{4}n} \cdot 5.0313e^{-j0.5005} = 5.0313e^{j(\frac{\pi}{4}n - 0.6005)}.
\end{aligned}$$