Instructions:

- Answer on this questionnaire
- Print your name
- Sign the pledge below
- Closed book and notes
- Three 8 1/2 x 11 sheets of paper allowed
- Calculators allowed
- Read the questions carefully.
- Problems 1 to 7 are multiple-choice. Each has 7 points. No partial credit will be given.
- In Problems 8 and 10, partial credit will be given. You must show your derivations/calculations to get full credit.

Name:
UMID:

PLEDGE: I have neither given nor received any aid on this exam, nor have I concealed any violations of the Honor Code.

SIGNATURE:

DO NOT TURN THIS PAGE OVER UNTIL TOLD TO DO SO!
Good Luck!

Mark your answers for Problems (1)–(7)

(1)	(a) (b) (c) (d) (e)
(2)	(a) (b) (c) (d) (e)
(3)	(a) (b) (c) (d) (e)
(4)	(a) (b) (c) (d) (e)
(5)	(a) (b) (c) (d) (e)
(6)	(a) (b) (c) (d) (e)
(7)	(a) (b) (c) (d) (e)
Subtotal	
(8)	
(9)	
(10)	
Total	

(1) A linear time-invariant system has the following signal relationship:

$$x[n] = 4\cos\left(\frac{\pi}{2}n + \frac{\pi}{3}\right) \mapsto y[n] = 8\cos\left(\frac{\pi}{2}n - \frac{\pi}{3}\right).$$

Then, the input signal $x[n] = \cos\left(\frac{\pi}{2}n + \frac{2\pi}{3}\right)$ will cause the system to produce

(a) $4\cos\left(\frac{\pi}{2}n - \frac{2\pi}{3}\right)$
(b) $2\cos\left(\frac{\pi}{2}n + \frac{2\pi}{3}\right)$
(c) $4\cos\left(\frac{\pi}{2}n\right)$
(d) $2\cos\left(\frac{\pi}{2}n\right)$
(e) None of the above

Answer: (d)

For a sinusoidal input $x[n] = A\cos(\hat{\omega}n + \phi)$ to an LTI system with $\mathcal{H}(\hat{\omega})$, the output

$$y[n] = A|\mathcal{H}(\hat{\omega})|\cos\left(\hat{\omega}n + \phi + \angle\mathcal{H}(\hat{\omega})\right).$$

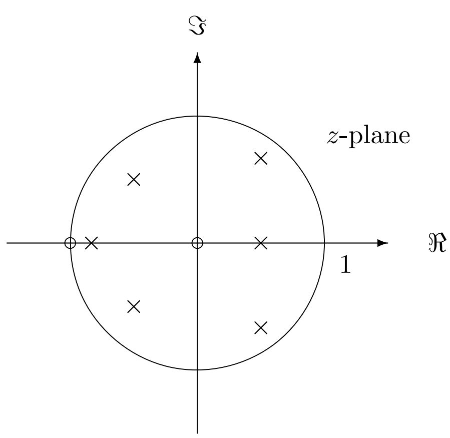
From the problem statement

$$\mathcal{H}(\frac{\pi}{2}) = 2\angle(-\frac{2\pi}{3}).$$

Therefore

$$x[n] = \cos\left(\frac{\pi}{2}n + \frac{2\pi}{3}\right) \mapsto y[n] = 2\cos\left(\frac{\pi}{2}n + \frac{2\pi}{3} - \frac{2\pi}{3}\right) = 2\cos\left(\frac{\pi}{2}n\right).$$

(2) An LTI filter with the following pole-zero diagram has the largest magnitude of the frequency response function $\mathcal{H}(\hat{\omega})$ at $\hat{\omega}$ equal to



(a) 0
(b) $\pi/3$
(c) $\pi/2$
(d) $2\pi/3$
(e) π

Answer: (b)

A peak occurs at the angle of poles with a larger magnitude of the frequency response function for a pole closer to the unit circle.

(3) For the following continuous-time signal

$$2\cos\left(2\pi 100t + \frac{\pi}{3}\right) - 3\sin\left(2\pi 400t - \frac{\pi}{6}\right) + 4\cos\left(2\pi 800t - \frac{\pi}{2}\right),$$

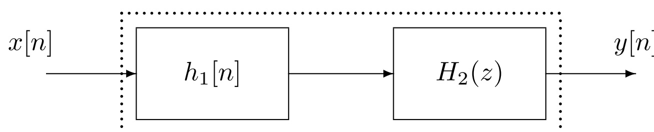
which of the following sampling frequencies (in samples per second) does not result in aliasing?

(a) 100
(b) 200
(c) 300
(d) 900
(e) $1,800$

Answer: (e)

The sampling frequency must exceed double the maximum frequency of the signal for no aliasing. Therefore, the sampling frequency must exceed $2 \times 800 = 1,600$ samples/sec.

(4) The overall impulse response $h[n]$ of the following two cascaded LTI filters



where $h_1[n] = \delta[n] + 2\delta[n-1]$ and $H_2(z) = 1 - 2z^{-2}$, is

(a) $\delta[n] - 4\delta[n-2]$
(b) $\delta[n] + 2\delta[n-1] - 2\delta[n-2]$
(c) $\delta[n] - 2\delta[n-2] - 2\delta[n-2]$
(d) $2\delta[n] + \delta[n-1] - 4\delta[n-2] - 2\delta[n-3]$
(e) $\delta[n] + 2\delta[n-1] - 2\delta[n-2] - 4\delta[n-3]$

Answer: (e)

The time-domain approach:

$$\begin{aligned} h[n] &= h_1[n] * h_2[n] = \big(\delta[n] + 2\delta[n-1]\big) * \big(\delta[n] - 2\delta[n-2]\big) \\ &= \delta[n] + 2\delta[n-1] - 2\delta[n-2] - 4\delta[n-3]. \end{aligned}$$

The z-domain approach:

$$\begin{aligned} H(z) &= H_1(z)H_2(z) = \big(1 + 2z^{-1}\big) \cdot \big(1 - 2z^{-2}\big) \\ &= 1 + 2z^{-1} - 2z^{-2} - 4z^{-3} \\ h[z] &= \delta[n] + 2\delta[n-1] - 2\delta[n-2] - 4\delta[n-3]. \end{aligned}$$

(5) The input-output relationship of an LTI filter is given as follows:

$$y[n] = x[n] + 2x[n-1].$$

A real periodic signal $x[n]$ with period 3 is applied to the filter. If the 3-point DFT of this signal is

$$X[0] = 0, \quad X[1] = j\sqrt{3}, \quad X[2] = -j\sqrt{3},$$

then the output $y[n]$ for $n = 0, 1, 2$ is

(a) $3\cos(\pi n/4)$
(b) $3\sin(\pi n/4)$
(c) $6\cos(2\pi n/3)$
(d) $6\sin(2\pi n/3)$
(e) None of the above

Answer: (c)

For a periodic signal $x[n]$ with period N,

$$y[n] = \sum_{k=0}^{N-1} Y[k] e^{j2\pi kn/N},$$

where

$$Y[k] = X[k]\mathcal{H}\left(\frac{2\pi k}{N}\right), \quad k = 0, \ldots, N-1.$$

We note that $N = 3$ and that

$$\begin{aligned}
\mathcal{H}\left(\frac{2\pi k}{N}\right) = H(z)\, \big|_{z=\exp(j\frac{2\pi k}{N})} &= 1 + e^{-j\frac{2\pi k}{N}} \\
&= \begin{cases} 2, k = 0, & \\ -j\sqrt{3}, & k = 1, \\ j\sqrt{3}, & k = 2. \end{cases}
\end{aligned}$$

Then

$$Y[k]X[k]\mathcal{H}\left(\frac{2\pi k}{N}\right) \qquad = \begin{cases} 0, k = 0, & \\ 3, & k = 1, \\ 3, & k = 2. \end{cases}$$

Therefore,

$$\begin{aligned}
y[n] &= \sum_{k=0}^{N-1} Y[k] e^{j2\pi kn/N} \\
&= 3 \cdot e^{j2\pi n/3} + 3 \cdot e^{j4\pi n/3} \\
&= 3 \cdot e^{j2\pi n/3} + 3 \cdot e^{j4\pi n/3 - 2\pi n} \\
&= 3 \cdot e^{j2\pi n/3} + 3 \cdot e^{-j2\pi n/3} = 6\cos(2\pi n/3).
\end{aligned}$$

(6) The z-transform of

$$x[n] = a^n u[n] + (-a)^n u[n]$$

is

(a) 0
(b) $2/(1 - a^2 z^{-2})$
(c) $2z^{-1}/(1 - a^2 z^{-2})$
(d) $2/(1 - az^{-1})^2$
(e) None of the above

Answer: (b)

$$\begin{aligned}
X(z) &= \frac{1}{1 - az^{-1}} + \frac{1}{1 + az^{-1}} \\
&= \frac{1 + az^{-1} + 1 - az^{-1}}{(1 - az^{-1})(1 + az^{-1})} \\
&= \frac{2}{1 - a^2 z^{-2}}.
\end{aligned}$$

(7) The inverse z-transform of

$$W(z) = \frac{1 + z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

is

(a) $w[n] = 6 \cdot (-\frac{1}{2})^n u[n] + 8 \cdot (-\frac{1}{3})^n u[n]$.
(b) $w[n] = 6 \cdot (\frac{1}{2})^n u[n] + 8 \cdot (\frac{1}{3})^n u[n]$.
(c) $w[n] = 6 \cdot (\frac{1}{2})^n u[n] - 8 \cdot (\frac{1}{3})^n u[n]$.
(d) $w[n] = 9 \cdot (\frac{1}{2})^n u[n] + 8 \cdot (\frac{1}{3})^n u[n]$.
(e) $w[n] = 9 \cdot (\frac{1}{2})^n u[n] - 8 \cdot (\frac{1}{3})^n u[n]$.

Answer: (e)

$$\begin{aligned} W(z) &= \frac{1 + z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \\ &= \frac{1 + z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})} \\ &= \frac{A}{(1 - \frac{1}{2}z^{-1})} + \frac{B}{(1 - \frac{1}{3}z^{-1})}, \end{aligned}$$

where

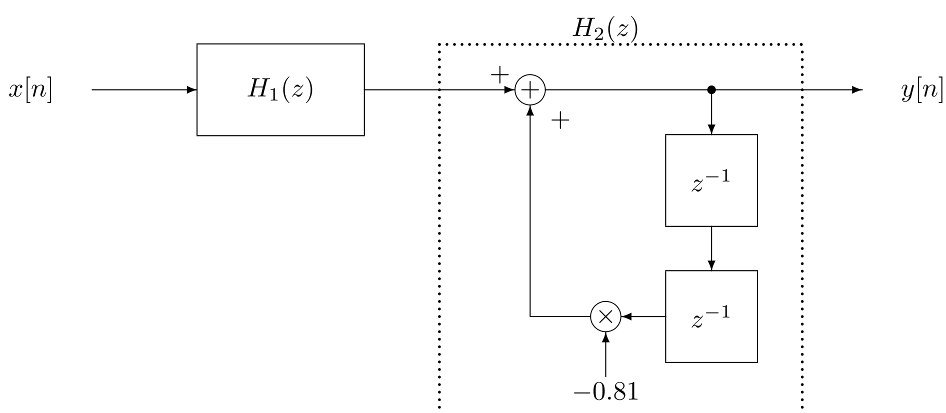
$$A = W(z)(1 - \frac{1}{2}z^{-1}) \Big|_{z^{-1}=2} = \frac{1 + z^{-1}}{(1 - \frac{1}{3}z^{-1})} \Big|_{z^{-1}=2} = 9,$$

$$B = W(z)(1 - \frac{1}{3}z^{-1}) \Big|_{z^{-1}=3} = \frac{1 + z^{-1}}{(1 - \frac{1}{2}z^{-1})} \Big|_{z^{-1}=3} = -8.$$

Therefore, $w[n] = 9 \cdot (\frac{1}{2})^n u[n] - 8 \cdot (\frac{1}{3})^n u[n]$.

(8) (16 points) Consider the following block diagram of the cascaded LTI filters, where

$$H_1(z) = 1 + z^{-1}.$$



(a) Find the overall system function $H(z)$.

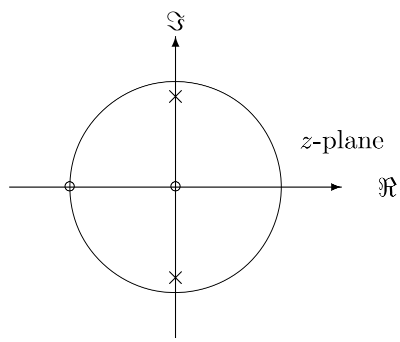
We note that $H_2(z) = \frac{1}{1+0.81z^{-2}}$.

$$H(z) = H_1(z)H_2(z) = \frac{1+z^{-1}}{1+0.81z^{-2}} = \frac{z^2+z}{z^2+0.81}.$$

(b) Plot the pole-zero diagram of $H(z)$.

$$\begin{aligned} H(z) = H_1(z)H_2(z) &= \frac{1+z^{-1}}{1+0.81z^{-2}} \\ &= \frac{z^2+z}{z^2+0.81} \\ &= \frac{z(z+1)}{(z+0.9j)(z-0.9j)} \end{aligned}$$

So $H(z)$ has zeros at 0 and -1 and poles at $\pm 0.9j$.

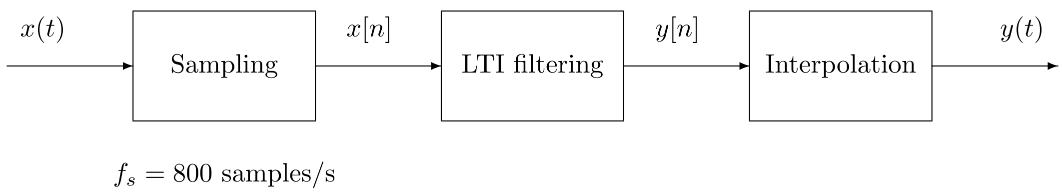


(9) (25 points) The following continuous-time signal is to be sampled, filtered and (ideally) reconstructed.

$$x(t) = 10 + 4\cos\big(2\pi 100t\big) + 2\cos\big(2\pi 200t + \frac{\pi}{2}\big).$$

Let the sampling frequency $f_s = 800$ samples/sec and $x[n] = x(nT_s) = x(n/f_s)$ and the LTI filter is described by the following difference equation.

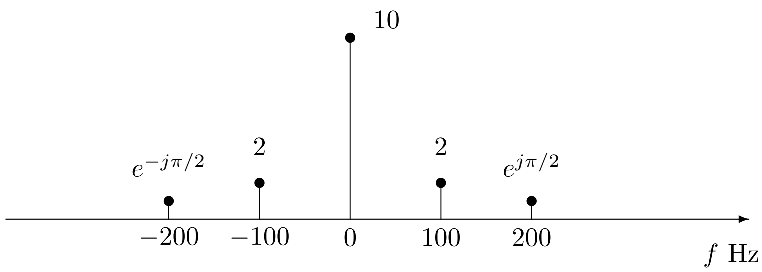
$$y[n] = x[n] - x[n-1] + x[n-2] - x[n-3].$$



(a) Plot the double-sided spectrum of $x(t)$ as a function of f Hz.

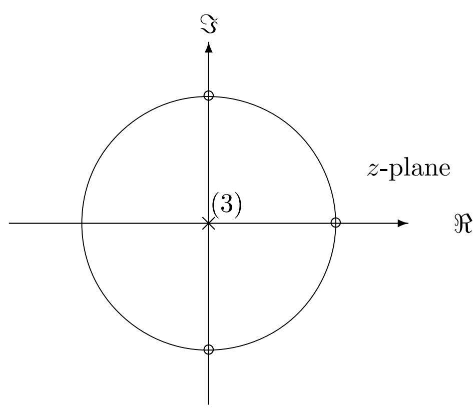
We note that, using the inverse Euler formula,

$$\begin{aligned}
x(t) &= 10 + 4\cos\big(2\pi 100t\big) + 2\cos\big(2\pi 200t + \frac{\pi}{2}\big) \\
&= 10 + 4\frac{e^{j2\pi 100t} + e^{-j2\pi 100t}}{2} + 2\frac{e^{j(2\pi 100t + \frac{\pi}{2})} + e^{-j(2\pi 100t + \frac{\pi}{2})}}{2} \\
&= 10 + 2e^{j2\pi 100t} + 2e^{j2\pi(-100)t} + e^{j\frac{\pi}{2}}e^{j2\pi 200t} + e^{-j\frac{\pi}{2}}e^{j2\pi(-200)t}.
\end{aligned}$$



(b) Find $H(z)$ and plot the pole-zero diagram.
From the difference equation

$$\begin{aligned}
H(z) &= 1 - z^{-1} + z^{-2} - z^{-3} \\
&= \frac{z^3 - z^2 + z - 1}{z^3} \\
&= \frac{z^2(z-1) + (z-1)}{z^3} = \frac{(z-1)(z^2+1)}{z^3} = \frac{(z-1)(z+j)(z-j)}{z^3}.
\end{aligned}$$



(c) Find and simplify $y[n]$ for $x[n]$ that is generated from sampling $x(t)$.
From the problem statement

$$\begin{aligned}
x[n] &= 10 + 4\cos\big(2\pi 100n/800\big) + 2\cos\big(2\pi 200n/800 + \frac{\pi}{2}\big) \\
&= 10 + 4\cos\big(\pi n/4\big) + 2\cos\big(\pi n/4 + \frac{\pi}{2}\big),
\end{aligned}$$

which is a sum of sinusoids. Then

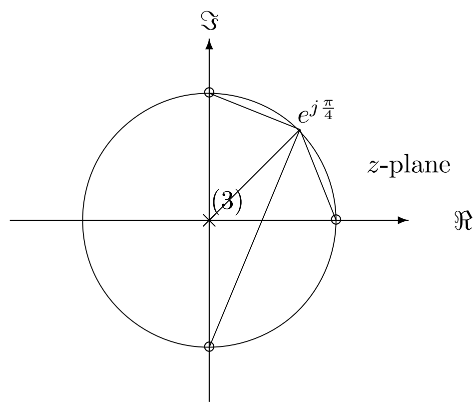
$$y[n] = \mathcal{H}(0)\cdot 10 + \mathcal{H}(\pi/4)\cdot 4\cos\big(\pi n/4\big) + \mathcal{H}(\pi/2)\cdot 2\cos\big(\pi n/4 + \frac{\pi}{2}\big),$$

where

$$
\begin{aligned}
\mathcal{H}(0) &= H(e^{j0}) = 0 \quad \text{due to the zero at } z = 1, \\
\mathcal{H}(\pi/2) &= H(e^{j\pi/2}) = 0 \quad \text{due to the zero at } z = e^{j\pi/2}, \\
\mathcal{H}(\pi/4) &= H(e^{j\pi/4}) = \frac{(z-1)(z+j)(z-j)}{z^3} \Big|_{z=e^{j\pi/4}} \\
&= \frac{0.765\angle 1.963 \cdot 1.848\angle 1.178 \cdot 0.765\angle(-0.392)}{1\angle(3\pi/4)} \\
&= 1.082\angle 0.392.
\end{aligned}
$$

Therefore,

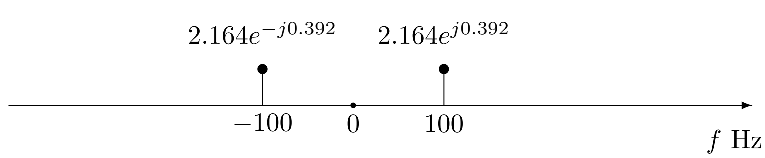
$$
\begin{aligned}
y[n] &= 0 \cdot 10 + 1.082 \cdot 4\cos\big(\pi n/4 + 0.392\big) + 0 \cdot 2\cos\big(\pi n/4 + \frac{\pi}{2}\big) \\
&= 4.328\cos\big(\pi n/4 + 0.392\big).
\end{aligned}
$$



(d) Find the reconstruction $y(t)$ from $y[n]$ using the ideal interpolation.
We note that the frequency $\hat{\omega} = \frac{\pi}{4}$ of $y[n]$ less than π. Then from ideal interpolation

$$
\begin{aligned}
y[n] = 4.328\cos\big(\frac{\pi}{4}n + 0.392\big) \implies y(t) &= 4.328\cos\big(\frac{\pi}{4} f_s \underbrace{nT_s}_{t} + 0.392\big) \\
&= 4.328\cos\big(\frac{\pi}{4} f_s t + 0.392\big) \\
&= 4.328\cos\big(2\pi 100 t + 0.392\big).
\end{aligned}
$$

(e) Plot the double-sided spectrum of $y(t)$ as a function of f Hz.



(10) (20 points) You are asked to design a second-order IIR filter whose system function $H(z)$ is of the form of

$$
H(z) = \frac{z+1}{(z - re^{j\theta})(z - re^{-j\theta})}.
$$

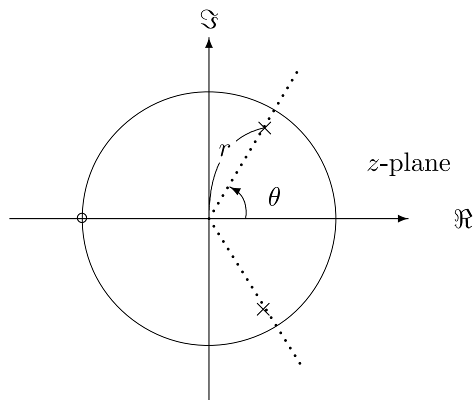
The designed filter must meet the following three constraints:

(i) It must be stable.

(ii) $|\mathcal{H}(\hat{\omega})|$ has a peak at $\hat{\omega} = \frac{\pi}{3}$.

(iii) $|\mathcal{H}(0)| = \frac{100}{42}$.

(a) Find the values for r and θ. (There are two possible values for r. Choose the one that gives a higher peak.)
Since a peak must be at $\hat{\omega} = \frac{\pi}{3}$, it must be that $\theta = \frac{\pi}{3}$. And the stability condition dictates that $r < 1$ (all the poles must reside inside of the unit circle centered at the origin.) With this information, the pole-zero diagram will look like the following.



Now at $\hat{\omega} = 0$

$$
\begin{aligned}
|\mathcal{H}(0)| = |H(e^{j0})| = |H(1)| &= \frac{1+1}{(1 - re^{j\frac{\pi}{3}})(1 - re^{-j\frac{\pi}{3}})} \\
&= \frac{2}{1 - 2r\cos(\frac{\pi}{3}) + r^2} \\
&= \frac{2}{1 - r + r^2} = \frac{100}{42},
\end{aligned}
$$

from which we have

$$
r^2 - r + 1 = \frac{2 \cdot 42}{100} = 0.84 \implies r^2 - r + 0.16 = 0.
$$

Solutions to the above quadratic equation are

$$
r = \frac{1 \pm \sqrt{1 - 4 \cdot 0.16}}{2} = \frac{1 \pm 0.6}{2} = 0.8 \text{ or } 0.2.
$$

Since the peak gets higher for a pole that is closer to the unit circle. The choice is $r = 0.8$.

(b) Find the resulting input/output difference equation .
From the previous part

$$
H(z) = \frac{z+1}{(z - 0.8e^{j\frac{\pi}{3}})(z - 0.8e^{-j\frac{\pi}{3}})} = \frac{z^{-1} + z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}},
$$

from which we get

$$
y[n] = 0.8y[n-1] - 0.64y[n-2] + x[n-1] + x[n-2].
$$

I wish each of you a rewarding summer!