1 Goals

1. Review of complex numbers
   (a) Definition and representations
   (b) Operations on complex numbers
2. Complex exponential signals
3. Reduction of a sum of sinusoids of the same frequency

2 Motivating Problem

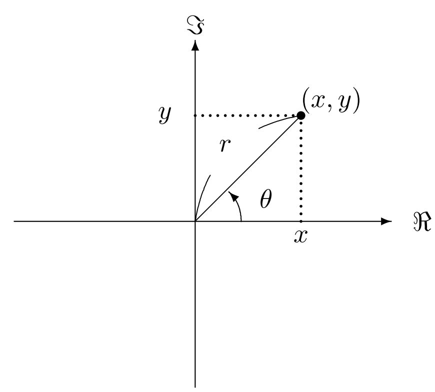
How do we reduce a sum of sinusoids of the same frequency to a single sinusoid?

Example 2.1

$$3\cos(2\pi 60t + \frac{\pi}{3}) + 4\cos(2\pi 60t - \frac{\pi}{6}) = 5\cos(2\pi 60t + 0.1199).$$

- (Complex number) The unit of imaginary numbers $j = \sqrt{-1}$.

$$z = x + jy = re^{j\theta} = r\angle\theta.$$
$$r = \sqrt{x^2+y^2}, \quad \theta \stackrel{?}{=} \tan^{-1}\frac{y}{x}.$$
$$x = r\cos\theta, \quad y = r\sin\theta.$$
$$x = \Re\{z\}, \quad y = \Im\{z\}.$$



- (Euler's formula)

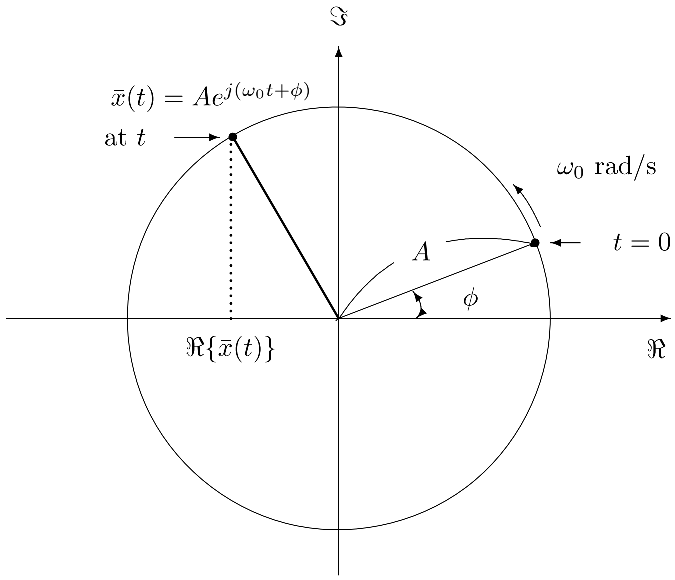
$$e^{j\theta} = \cos\theta + j\sin\theta.$$

- (Complex Exponential Signal or Rotating Phasor)

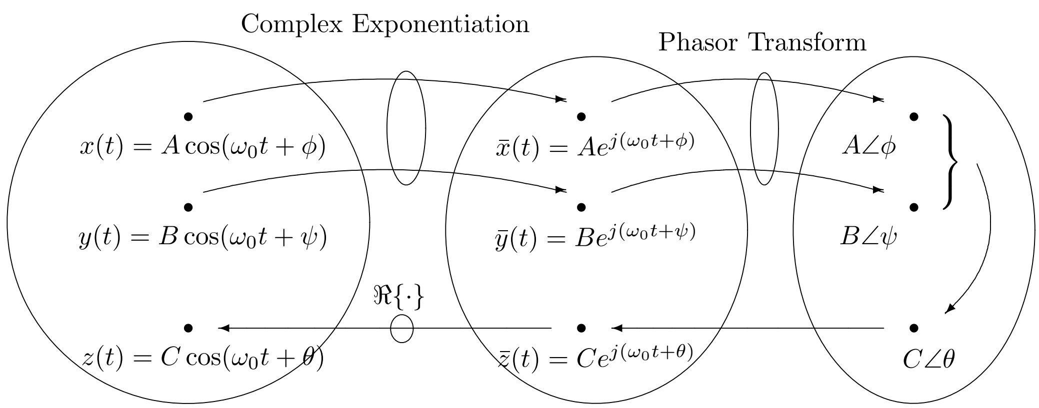
$$\bar{x}(t) = Ae^{j\left(\omega_0 t+\phi\right)} = \underbrace{A\cos\left(\omega_0 t+\phi\right)}_{\text{real part}} + j\underbrace{A\sin\left(\omega_0 t+\phi\right)}_{\text{imaginary part}}.$$

- (Real part of a complex exponential signal)

$$\bar{x}(t) = Ae^{j\left(\omega_0 t+\phi\right)} \implies \Re\{\bar{x}(t)\} = A\cos\left(\omega_0 t+\phi\right).$$



- Synopsis of sinusoid reduction



(Step 1) Find corresponding complex exponential signals (rotating phasors).

(Step 2) Find corresponding complex amplitudes (phasors).

(Step 3) Compute with complex numbers.

(Step 4) Transform the resulting complex amplitude (phasor) back to a complex exponential signal.

(Step 5) Take the real part to get the sinusoid.

- (Example Worked)

(a) Sinusoids are the real part of complex exponentials.

$$3\cos(2\pi 60t + \frac{\pi}{3}) = \Re\{3e^{j(2\pi 60t + \frac{\pi}{3})}\}$$
$$4\cos(2\pi 60t - \frac{\pi}{6}) = \Re\{4e^{j(2\pi 60t - \frac{\pi}{6})}\}$$

$$z(t) = 3\cos(2\pi 60t + \frac{\pi}{3}) + 4\cos(2\pi 60t - \frac{\pi}{6}) = \Re\{3e^{j(2\pi 60t + \frac{\pi}{3})}\} + \Re\{4e^{j(2\pi 60t - \frac{\pi}{6})}\}.$$

(b) Sum of complex exponentials with the same frequency

$$\begin{aligned} z(t) = \Re\{3e^{j(2\pi 60t + \frac{\pi}{3})}\} + \Re\{4e^{j(2\pi 60t - \frac{\pi}{6})}\} &= \Re\{3e^{j(2\pi 60t + \frac{\pi}{3})} + 4e^{j(2\pi 60t - \frac{\pi}{6})}\} \\ &= \Re\Big\{e^{j2\pi 60t} \underbrace{(3e^{j\frac{\pi}{3}} + 4e^{-j\frac{\pi}{6}})}_{\text{complex number } z}\Big\} \end{aligned}$$

(c) Operations on complex numbers

$$\begin{aligned} z &= 3e^{j\frac{\pi}{3}} + 4e^{-j\frac{\pi}{6}} \\ &= \Big(3\cos\frac{\pi}{3} + j3\sin\frac{\pi}{3}\Big) + \Big(4\cos(-\frac{\pi}{6}) + j\sin(-\frac{\pi}{6})\Big) \\ &= (1.5000 + j2.5981) + (3.4641 - j2.0000) \\ &= 4.9641 + j0.5981 \\ &= 5\angle 0.1199 = 5e^{j0.1199}. \end{aligned}$$

(d) Result

$$\begin{aligned} z(t) &= \Re\{e^{j2\pi 60t} \cdot 5e^{j0.1199}\} \\ &= \Re\{5e^{j(2\pi 60t + 0.1199)}\} \\ &= 5\cos(2\pi 60t + 0.1199). \end{aligned}$$

- (Conclusion) A sum of sinusoids of the same frequency can be simplified using operations on complex numbers.

3 Complex Numbers

1. (Definition) A **complex** number z is any number that can be expressed in the form of

$$z = x + jy, \quad \text{where } x \text{ and } y \text{ are real numbers.}$$

   (a) The real part of z: $x = \Re\{z\}$.

   (b) The imaginary part of z: $y = \Im\{z\}$.

   (c) Note that both parts are real.

2. (Equality) Two complex numbers z_1 and z_2 are equal if and only if

$$\Re\{z_1\} = \Re\{z_2\} \quad \text{and} \quad \Im\{z_1\} = \Im\{z_2\}$$

3. (Representations) There are three ways of representing a complex number z.

(a) Rectangular form: $z = x + jy$

(b) Exponential form: $z = re^{\theta}$

(c) Polar form: $z = r\angle\theta$

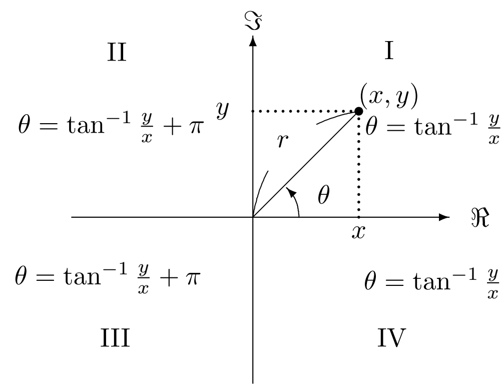
$$r = |z| = \text{ the magnitude or length or modulus of } z,$$
$$\theta = \arg z = \text{ the angle of } z.$$

4. (Conversion between forms)

   (a) $z = x + jy \implies re^{j\theta}$ or $r\angle\theta$: find r and θ.

   $$r = \sqrt{x^2 + y^2},$$
   $$\theta = \tan^{-1}\frac{y}{x} + \textcircled{\pi}$$

   Wanring: Most calculators can *not* handle automatically the angle in the 2nd and 3rd quadrants.

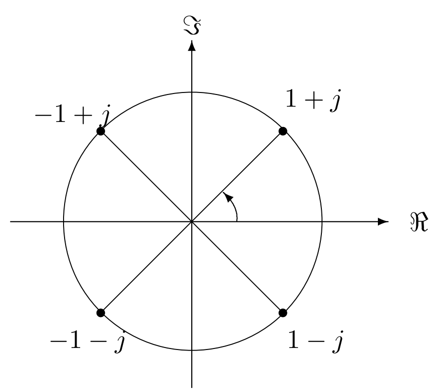


   (b) $re^{j\theta}$ or $r\angle\theta \implies z = x + jy$: find x and y.

   $$x = r\cos\theta,$$
   $$y = r\sin\theta.$$

Example 3.1 Express the following complex numbers in the exponential form.

$$z_1 = 1 + j, \quad z_2 = -1 + j, \quad z_3 = -1 - j, \quad z_4 = 1 - j.$$



3.1 Complex Arithmetic

(a) Addition and subtraction: use the rectangular form

(b) Multiplication and division: use the exponential or polar form

(c) Complex conjugation

(d) Complex exponentials

3.1.1 Addition

$$z = x + jy, \quad w = u + jv \quad \implies \quad z + w = (x + u) + j(y + v).$$

3.1.2 Multiplication

$$z = x + jy, \quad w = u + jv \quad \implies \quad zw = (x + jy)(u + jv) = (xu - yv) + j(yu + xv).$$

Multiplication is best done in the exponential or polar form.

$$z = re^{j\theta}, \quad w = \rho e^{j\phi}, \quad \implies \quad zw = r\rho e^{j(\theta+\phi)} = r\rho\angle(\theta + \phi).$$

Or

$$|zw| = |z|\,|w|,$$
$$\arg zw = \arg z + \arg w.$$

3.1.3 Conjugation

(a) (Definition) Given $z = x + jy$, its complex conjugate is defined to be

$$z^* = x - jy.$$

(b) (Facts)

(a) Exponential or polar form

$$z = re^{j\theta} \implies z^* = re^{-j\theta}.$$
$$|z| = |z^*|, \quad \arg z = -\arg z^*.$$

(b) $zz^* = |z|^2$

(c) $2\Re\{z\} = z + z*$

(d) $j2\Im\{z\} = z - z^*$.

3.1.4 Complex Exponentials

1. Given $z = x + jy$, what is e^z?
$$e^z = e^{x+jy} = e^x e^{jy}.$$

2. What is e^{jy}? Euler's formula states
$$e^{jy} = \cos y + j \sin y.$$

   Note that

$$|e^{jy}| = \sqrt{\cos^2 y + \sin^2 y} = 1,$$
$$\arg e^{jy} = \tan^{-1}\frac{\sin y}{\cos y} = \tan^{-1}\big(\tan(y)\big) = y.$$

3. Combining

$$|e^z| = |e^x e^{jy}| = |e^x||e^{jy}| = e^x,$$
$$\arg e^z = y.$$