1 Goals

1. Complex signals
   (a) Definition and graphs
   (b) Properties
2. Complex exponential signals: Revisit
   (a) Sinusoids of the same frequency
   (b) Sinusoids of differing frequencies
      i. Beat notes
      ii. Spectrum

2 Complex Signals

2.1 Definition and Graphs

A complex signal $z(t)$ or $z[n]$ is a signal of the following form:

$$z(t) = \underbrace{x(t)}_{\Re\{\}} + j \underbrace{y(t)}_{\Im\{\}},$$
$$z[n] = x[n] + jy[n].$$

Example 2.1

$$z(t) = 3e^{-2t}\cos(\omega_0 t) + j4e^{-2t}\sin(\omega_0 t).$$
$$z[n] = (\frac{1}{2})^n - j(\frac{1}{3})^n.$$

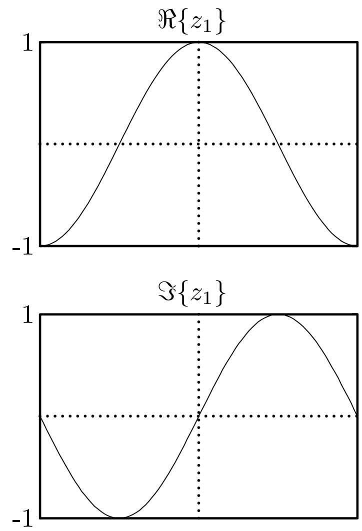
Plotting Graphs of a Complex Signal There are three ways of plotting a complex signal.

1. Plot the real and imaginary parts separately.
2. Plot the magnitude and phase of
$$z(t) = \big|z(t)\big|e^{j\arg z(t)}$$
   (i) separately, or (ii) together in the complex plane.
3. Plot it as a 3-D graph—a spiral plot

Example 2.2 Let $z_n(t) = e^{j2\pi nt}$ on $[-\frac{1}{2}, \frac{1}{2}]$.

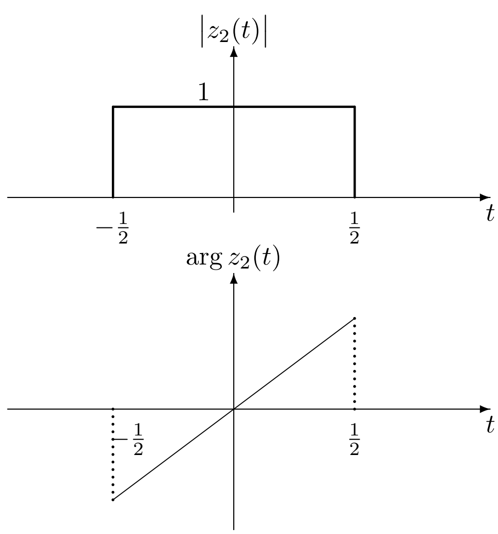
(a) Plot the real and imaginary parts of $z_1(t)$.
Using Euler's formula, we get

$$z_1(t) = e^{j2\pi t} = \cos(2\pi t) + j\sin(2\pi t).$$



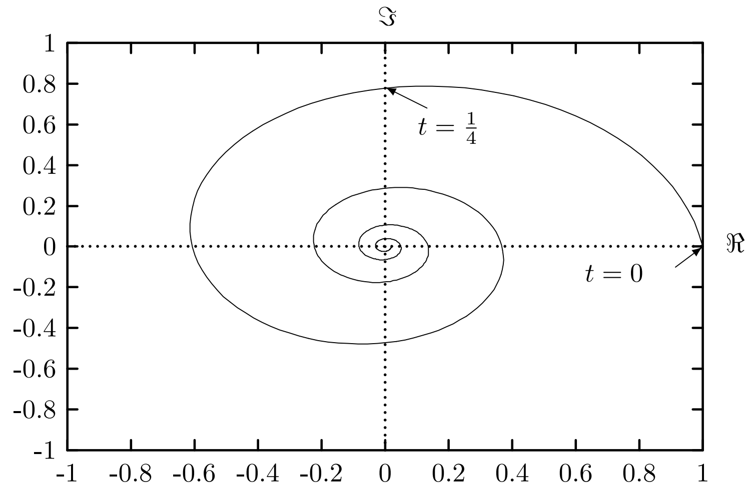
(b) Plot the magnitude and phase of $z_2(t)$ separately.

$$\begin{aligned}\left|z_2(t)\right| &= \left|e^{j2\pi 2t}\right| = 1.\\ \arg z_2(t) &= 2\pi 2t.\end{aligned}$$

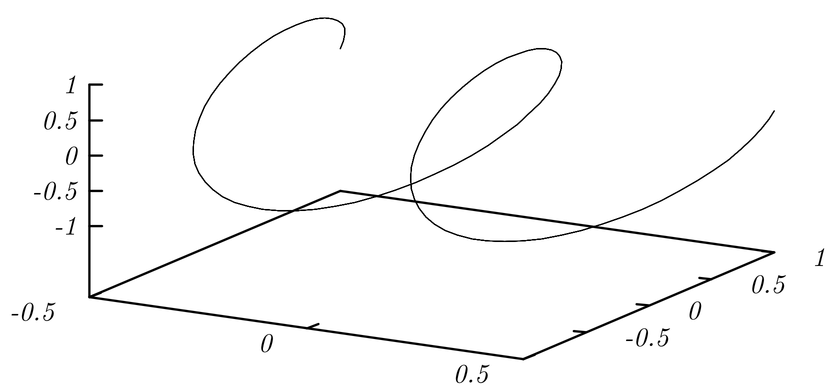


(c) Plot the magnitude and phase of complex signal $z(t) = e^{(-1+j2\pi)t}$ in the complex plane for $t \in [0, 4]$.

$$z(t) = e^{-t}e^{j2\pi t} \quad \Longrightarrow \quad \left|z(t)\right| = e^{-t}, \quad \arg z(t) = 2\pi t.$$



(d) Plot $z_2(t)$ as a 3-D plot



2.2 Properties of Complex Signals

(a) (Support Interval and Duration) The support interval of a complex signal $z(t)$ is the smallest interval outside which

$$z(t) = 0.$$

Note that $z(t) = 0 \iff x(t) = 0$ and $y(t) = 0$.

The duration of $z(t)$ is the size of its support interval.

(b) (Mean Value) $z(t) = x(t) + jy(t)$.

$$\begin{aligned}
M(z) &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} z(t)\, dt \\
&= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x(t)\, dt + j \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} y(t)\, dt \\
&= M(x) + jM(y).
\end{aligned}$$

(c) (Instantaneous Power)

$$|z(t)|^2 = z(t)z^*(t) = x^2(t) + y^2(t).$$

(d) (Average Power or Mean-Square Value)

$$\begin{aligned}
MS(z) &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |z(t)|^2\, dt \\
&= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(x^2(t) + y^2(t)\right) dt \\
&= MS(x) + MS(y).
\end{aligned}$$

(e) (Energy)

$$\begin{aligned}
E(z) &= \int_{t_1}^{t_2} |z(t)|^2\, dt \\
&= \int_{t_1}^{t_2} \left(x^2(t) + y^2(t)\right) dt \\
&= E(x) + E(y).
\end{aligned}$$

2.3 Correlation of Complex Signals

The **correlation** between two complex signals $z_1(t)$ and $z_2(t)$ is defined by

$$C(z_1, z_2) = \int_{t_1}^{t_2} z_1(t) z_2^*(t)\, dt.$$

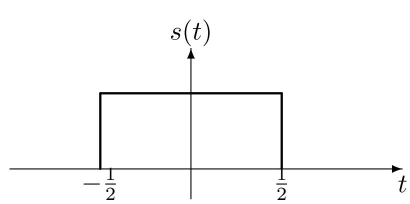
(a) $C(z_1, z_2)$ can be a complex number.

(b) (Warning) $C(z_1, z_2)$ is order-sensitive. In general, $C(z_1, z_2) = C^*(z_2, z_1)$.

(c) Energy $E(z)$

$$E(z) = \int_{t_1}^{t_2} z(t)z^*(t)\, dt = \int_{t_1}^{t_2} |z(t)|^2\, dt.$$

Example 2.3 Let $z_n(t) = e^{j2\pi nt}$, $n = 1, 2, \ldots$ on $[-\frac{1}{2}, \frac{1}{2}]$. A signal $s(t)$ is given below.



(a) Find $C(s, z_n)$.

$$
\begin{aligned}
C(s, z_n) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} s(t) z_n^*(t)\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi nt}\, dt \\
&= \frac{1}{-j2\pi n} e^{-j2\pi nt} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{1}{-j2\pi} \big(e^{-j\pi n} - e^{j\pi n} \big) \\
&= \frac{1}{-j2\pi} \big(\cos(-\pi n) + j\sin(-\pi n) - \cos(\pi n) - j\sin(\pi n) \big) \\
&= 0
\end{aligned}
$$

(b) Find $C(u, z_n)$, where

$$
u(t) = \begin{cases} t, & |t| < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}
$$

From the definition

$$
\begin{aligned}
C(u, z_n) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} u(t) z_n^*(t)\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} t e^{-j2\pi nt}\, dt \\
&= \Big(\frac{1}{-j2\pi n} t e^{-j2\pi nt} - \frac{1}{(-j2\pi n)^2} e^{-j2\pi nt} \Big) \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{(-1)^n}{2\pi n} j.
\end{aligned}
$$

We note that

$$
\int t e^{at}\, dt = \frac{1}{a} t e^{at} - \frac{1}{a^2} e^{at} + c.
$$

3 Complex Exponentials: Revisit

3.1 Sinusoids and Complex Exponentials

$$
A\cos(\omega_0 t + \phi) \quad \Longleftrightarrow \quad \underbrace{Ae^{j(\omega_0 t + \phi)}}_{\text{rotating phasor}} = Ae^{j\phi} e^{j\omega_0 t} \quad \Longleftrightarrow \quad \underbrace{Ae^{j\phi}}_{\text{phasor}} \quad \text{with suppressed } \omega_0
$$

(a) Inverse Euler's formulas:

$$
\begin{aligned}
\cos\theta &= \frac{e^{j\theta} + e^{-j\theta}}{2}, \\
\sin\theta &= \frac{e^{j\theta} - e^{-j\theta}}{j2}.
\end{aligned}
$$

from

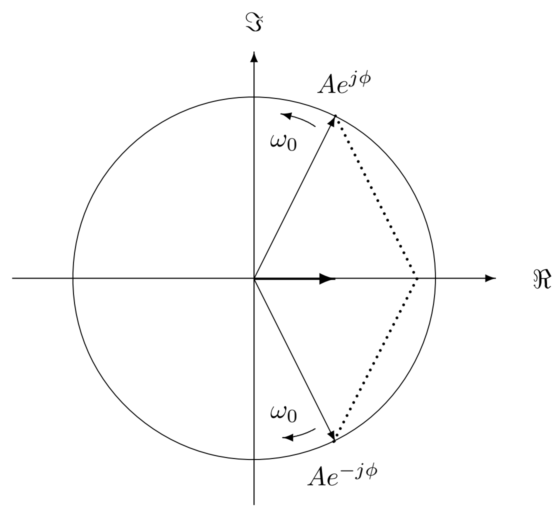
$$
\begin{aligned}
e^{j\theta} &= \cos\theta + j\sin\theta, \\
e^{-j\theta} &= \cos\theta - j\sin\theta.
\end{aligned}
$$

(b) So, $x(t) = A\cos(\omega_0 t + \phi)$ can be thought of as

(i) the real part of $\bar{x}(t)$

(ii) the sum of two rotating phasors that rotate in the opposite directions

$$A\cos(\omega_0 t + \phi) = \frac{Ae^{j(\omega_0 t+\phi)} + Ae^{-j(\omega_0 t+\phi)}}{2}.$$



(c) The sum of sinusoids of the same frequency can be found through the sum of corresponding phasors.

$$\sum_{k=1}^{K} A_k \cos(2\pi f_0 t + \phi_k) = C\cos(2\pi f_0 t + \theta),$$

where

$$Ce^{j\theta} = \sum_{k=1}^{K} A_k e^{j\phi_k}.$$

(d) (Negative frequencies)

(i) Sinusoids do not have negative frequencies by definition.

(ii) Complex exponentials (rotating phasors) do. Negative frequencies mean that the phasors rotate clockwise.

3.2 Complex Exponentials with Differing Frequencies

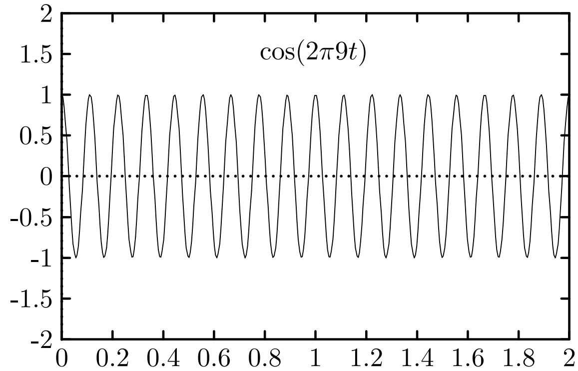
Are complex exponentials useful for summing sinusoids of differing frequencies?

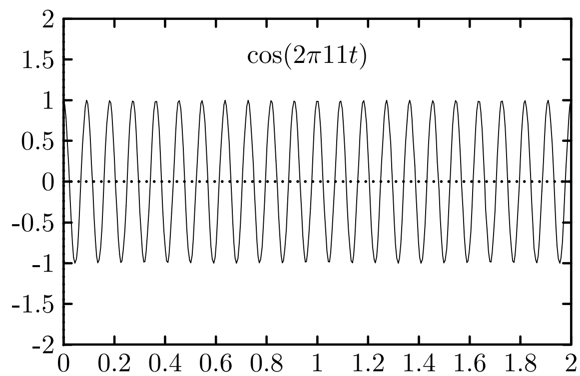
(a) Yes, in the case of two closely related frequencies

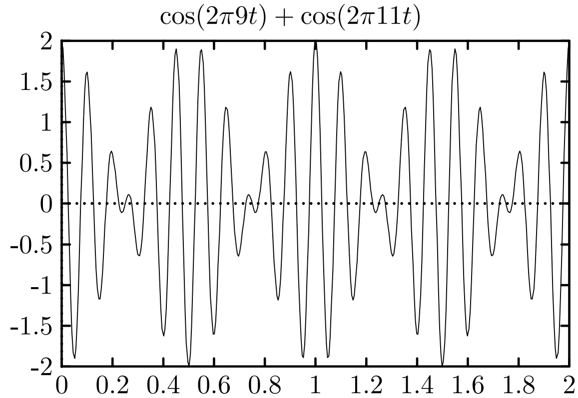
Example: Beat notes

(b) Yes, in the computation of the spectrum of a (not necessarily sinusoidal) signal.

Example 3.1 A beat note is a phenomenon, in which a tone signal (a sinusoid with a single frequency) shows fading-in/out loudness. This is due to constructive and destructive interferences of the two signals of similar frequencies. (So, the signal is not a tone—not a pure sinusoid, but rather two or more interfering tones.)



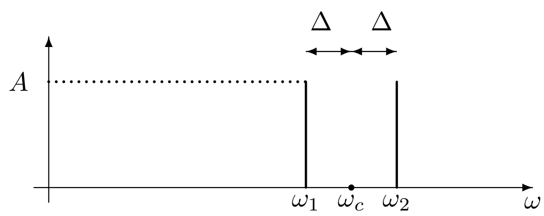




Analysis

$$
\begin{aligned}
x(t) &= A\cos(\omega_1 t) + A\cos(\omega_2 t) \\
&= 2A\cos(\Delta t)\cos(\omega_c t).
\end{aligned}
$$

When Δ is small, we hear ω_c fading in and out.



Note that $\omega_1 = \omega_c - \Delta$ and $\omega_2 = \omega_c + \Delta$.

$$\begin{aligned}
x(t) &= A\cos(\omega_1 t) + A\cos(\omega_2 t) \\
&= \Re\big\{Ae^{j\omega_1 t} + Ae^{j\omega_2 t}\big\} \\
&= A\Re\big\{e^{j(\omega_c-\Delta)t} + e^{j(\omega_c+\Delta)t}\big\} \\
&= A\Re\big\{e^{j\omega_c t}\big(e^{-j\Delta t} + e^{j\Delta t}\big)\big\} \\
&= A\Re\big\{e^{j\omega_c t} 2\cos(\Delta t)\big\} \\
&= 2A\cos(\Delta t)\cos(\omega_c t).
\end{aligned}$$