Reading: 5.1–5.6

Issues in Studying Filters

- Filter Description
  - (a) Types of filters
    - (i) FIR
    - (ii) IIR
  - (b) Ways of describing filters
    - (i) Filter coefficients of a difference equation
    - (ii) Impulse response
    - (iii) Block diagram
    - (iv) Transfer function (frequency response)
    - (v) System function
  - (c) Ways of describing the input signal
    - (i) time-domain
    - (ii) frequency-domain
    - (iii) z-domain
  - (d) Ways of describing the input/output relationship
    - (i) difference equation
    - (ii) convolution
    - (iii) transfer function (frequency response)
    - (iv) system function
- Filter Design
  - (a) Types of filters
    - (i) FIR
    - (ii) IIR
  - (b) Better design techniques: poles and zeros in the z-domain

1 LTI Filters

1.1 General Form of LTI Filters

(a) Filter Description: $\{b_k\}$

(b) Input-output relationship by the difference equation:

$$y[n] = \sum_{k=-M_1}^{M_2} b_k x[n-k],$$

where M_1 and M_2 are integers

$$0 \leq M_1, M_2 \leq \infty.$$

The order of the filter: $M_1 + M_2$.

Classification:

(a) FIR or IIR

  (i) FIR: $M_1, M_2 < \infty$.

  (ii) IIR: M_1 or $M_2 = \infty$.

(b) Causal or noncausal

  (i) Causal: $M_1 = 0$.

  (ii) Noncausal: $M_1 > 0$.

1.2 Example

An FIR filter is given by the following difference equation:

$$y[n] = x[n-2] + x[n-1] + 2x[n].$$

(a) Comparing with the general form

$$y[n] = \sum_{k=-M_1}^{M_2} b_k x[n-k],$$

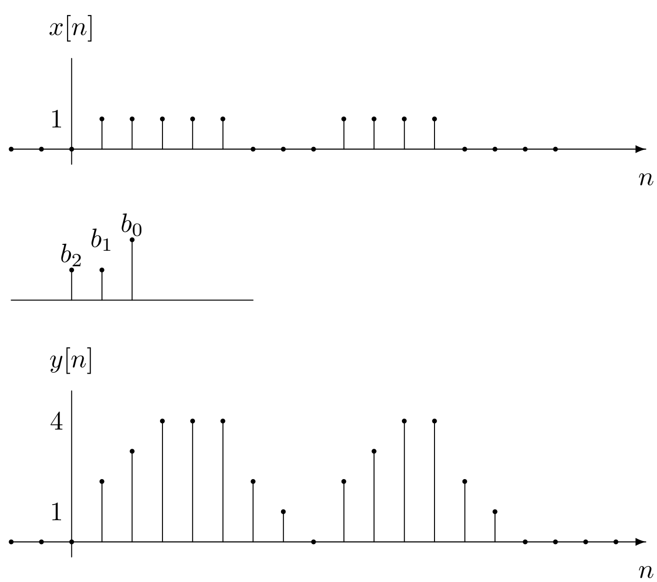
we have $M_1 = 0$, $M_2 = 2$,

$$b_0 = 2, b_1 = 1, b_2 = 1.$$

The order of the filter is $M_1 + M_2 = 2$.

(b) Note that it is causal.

(c) For a specific input signal $x[n]$



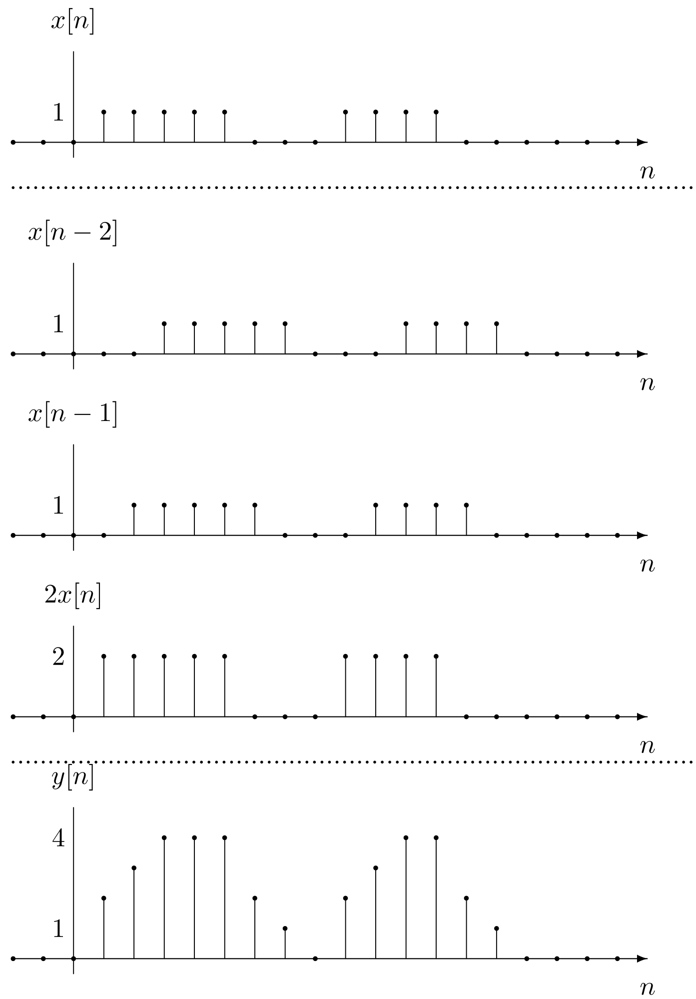
(d) Things to note

  (i) (**View 1**) This is a weighted running sum of the present and past input values.

(ii) The support of the output is longer than the input by the order.
(iii) The output is smoother. It is a kind of lowpass filter.
(iv) There are transient regions.
(v) The output is delayed.

(e) (**View 2**) $y[n]$ is a weighted sum of signals.

$$y[n] = x[n-2] + x[n-1] + 2x[n].$$



(f) (**View 3**) $y[n]$ is the convolution of the input and the impulse response of the filter.

2 Linearity and Time-Invariance

Filters described by

$$y[n] = \sum_{k=-M_1}^{M_2} b_k x[n-k],$$

are linear and time-invariant.

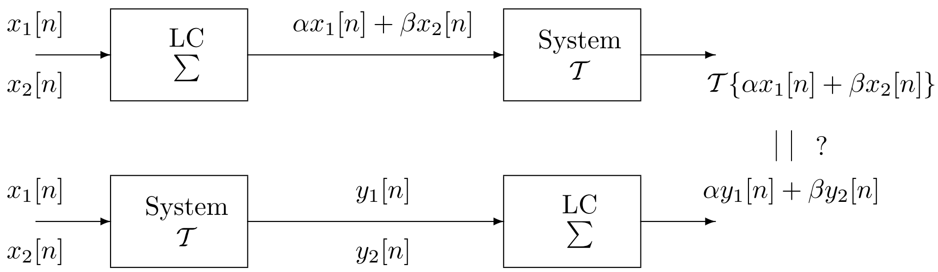
2.1 Linearity

Definition A system is linear if for any input signals $x_1[n]$ and $x_2[n]$ and any scalars α and β

$$\alpha x_1[n] + \beta x_2[n] \mapsto \alpha y_1[n] + \beta y_2[n],$$

where $x_1[n] \mapsto y_1[n]$ and $x_2[n] \mapsto y_2[n]$.

(Additive and scaling properties; or superposition principle)



Fact Filters described by $y[n] = \sum_{k=-M_1}^{M_2} b_k x[n-k]$ are linear.

Proof:

$$\begin{aligned}
\alpha x_1[n] + \beta x_2[n] &\mapsto \sum_{k=-M_1}^{M_2} b_k \big(\alpha x_1[n-k] + \beta x_2[n-k]\big) \\
&= \sum_{k=-M_1}^{M_2} b_k \alpha x_1[n-k] + \sum_{k=-M_1}^{M_2} b_k \beta x_2[n-k] \\
&= \alpha \sum_{k=-M_1}^{M_2} b_k x_1[n-k] + \beta \sum_{k=-M_1}^{M_2} b_k x_2[n-k] \\
&= \alpha y_1[n] + \beta y_2[n].
\end{aligned}$$

Examples of nonlinear systems

(a) $y[n] = x^2[n]$.

(b) $y[n] = \big|x[n]\big|$.

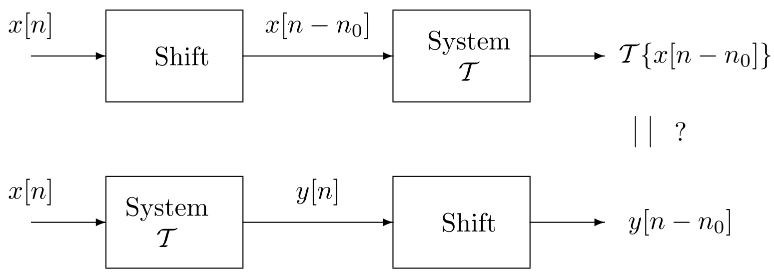
(c) $y[n] = x[n] + 3$.

2.2 Time-invariance

Definition A system is time-invariant if for any input signal $x[n]$ and any n_0

$$x[n-n_0] \mapsto y[n-n_0],$$

where $x[n] \mapsto y[n]$.



Fact Filters described by $y[n] = \sum_{k=-M_1}^{M_2} b_k x[n-k]$ are time-invariant.

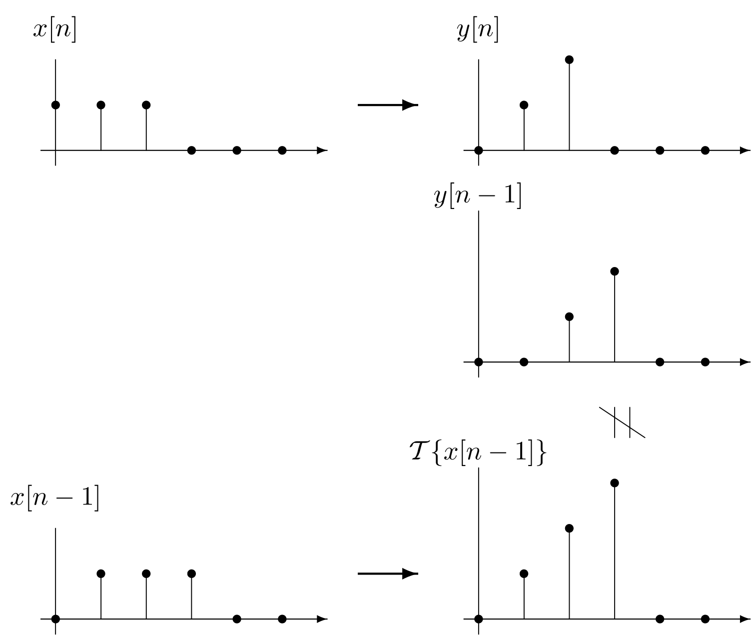
Proof: Let $x[n]$ and n_0 be given. Let $z[n] = x[n-n_0]$ be the input. Then the corresponding output is

$$\begin{aligned}\sum_{k=-M_1}^{M_2} b_k z[n-k] &= \sum_{k=-M_1}^{M_2} b_k x[n-k-n_0] \\ &= \sum_{k=-M_1}^{M_2} b_k x[(n-n_0)-k] \\ &= y[n-n_0].\end{aligned}$$

Hence, $x[n-n_0] \mapsto y[n-n_0]$.

Example of a time-varying system

$$y[n] = nx[n].$$



2.3 View 3—Convolution with the impulse response

View 3 An LTI filter performs convolution of the input with the impulse response of the filter. This viewpoint

(a) (Input) decomposes $x[n]$ into the sum of simple signals called *impulses*

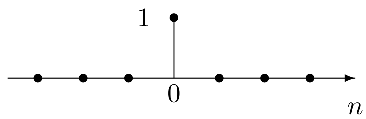
(b) (Filter) identifies the response of the filter to an impulse

(c) (Output) uses linearity and time-invariance to produce the output.

2.3.1 The unit impulse and the impulse response

- The unit impulse (sequence) or (Kronecker) delta function $\delta[n]$

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$



- The impulse response $h[n]$ (of an LTI system) is the output $y[n]$ when input $x[n] = \delta[n]$.

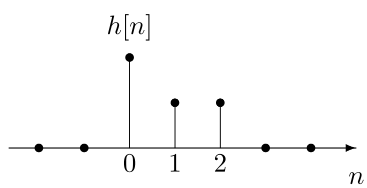
  The filters described by $y[n] = \sum_{k=-M_1}^{M_2} b_k x[n-k]$ has impulse response $h[n]$ given by

$$\begin{aligned} h[n] &= \sum_{k=-M_1}^{M_2} b_k x[n-k] \Big|_{x[n]=\delta[n]} \\ &= \sum_{k=-M_1}^{M_2} b_k \delta[n-k] \\ &= \begin{cases} b_n, & -M_1 \leq n \leq M_2, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Example An FIR filter is given by the following difference equation:

$$y[n] = x[n-2] + x[n-1] + 2x[n].$$

Find the impulse response $h[n]$ of the filter.

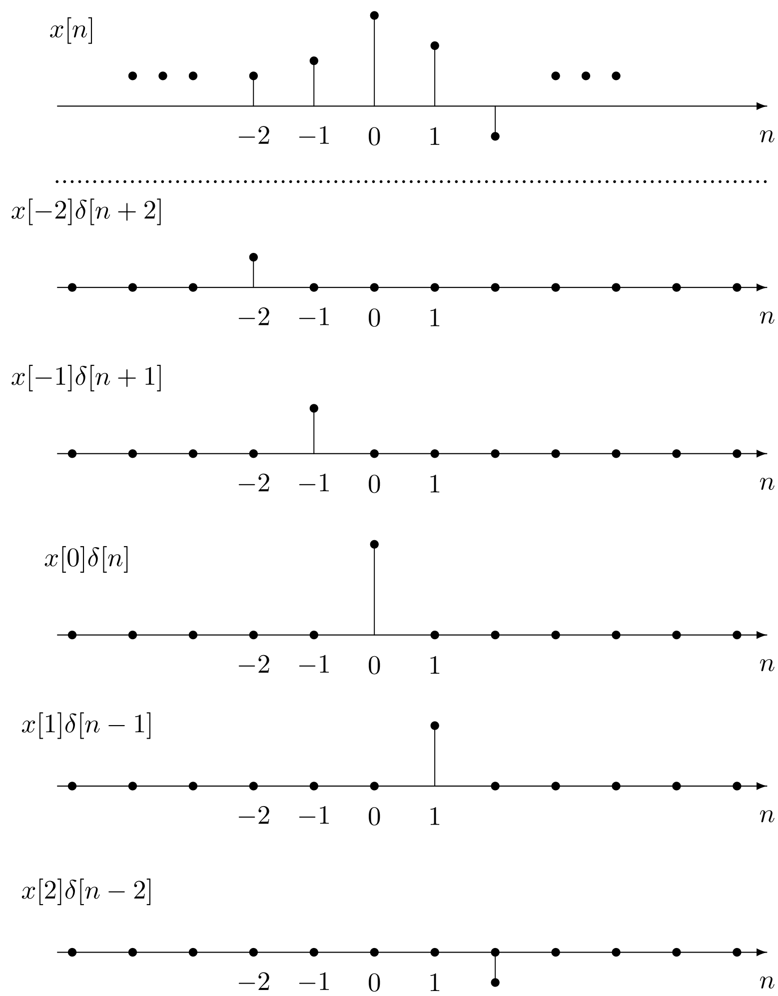


2.3.2 Decomposition of the input into impulses

For any signal $x[n]$

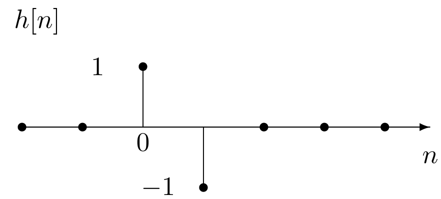
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots .$$

This decomposition shows that $x[n]$ can be viewed as the weighting of impulse functions.

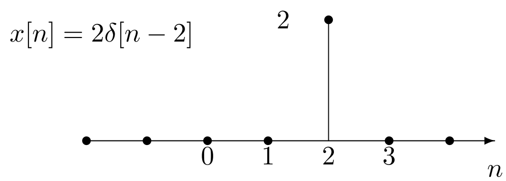


2.3.3 Description of the output with the impulse response

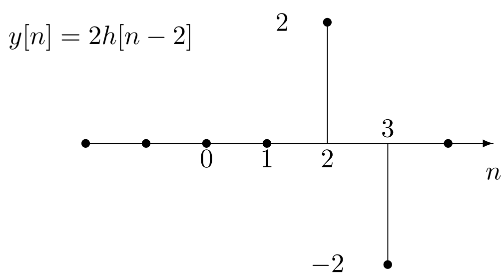
Example 1 An LTI filter has the following impulse response.



Find the response of the filter to the following input.



By linearity and time-invariance



□

An LTI filter with the impulse response $h[n]$ has the response $y[n]$ to input $x[n]$

$$x[n] \quad \mapsto \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Or

$$\delta[n] \mapsto h[n] \qquad \Longrightarrow \qquad x[n] \mapsto y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Fact

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-M_1}^{M_2} b_k x[n-k] \quad \text{for} \quad h[n] = \begin{cases} b_n, & -M_1 \le n \le M_2, \\ 0, & \text{else.} \end{cases} \end{aligned}$$