# 1 Convolution

## 1.1 Definition

The convolution of two signals $v[n]$ and $w[n]$ is defined to be

$$v[n] * w[n] = \sum_{k=-\infty}^{\infty} v[k]w[n-k] = \sum_{k=-\infty}^{\infty} w[k]v[n-k].$$

- Convolution is defined for arbitrary signals.
- It appears in other situations than filtering.
- There is a continuous-time version too.
- For our purpose, it is useful because of the following:

  An LTI filter with the impulse response $h[n]$ has the response $y[n]$ to input $x[n]$ given as follows.
  $$y[n] = x[n] * h[n] = h[n] * x[n].$$

- Computation of convolution

$$y[n] = h[n] * x[n] = \cdots + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + \cdots$$

$$\begin{aligned} y[n] = h[n] * x[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \cdots \\ &\quad + h[-1]x[n+1] \\ &\quad + h[0]x[n] \\ &\quad + h[1]x[n-1] \\ &\quad + \cdots \end{aligned}$$

$$\begin{aligned} y[n] = x[n] * h[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \cdots \\ &\quad + x[-1]h[n+1] \\ &\quad + x[0]h[n] \\ &\quad + x[1]h[n-1] \\ &\quad + \cdots \end{aligned}$$

## 1.2 Example of Convolution

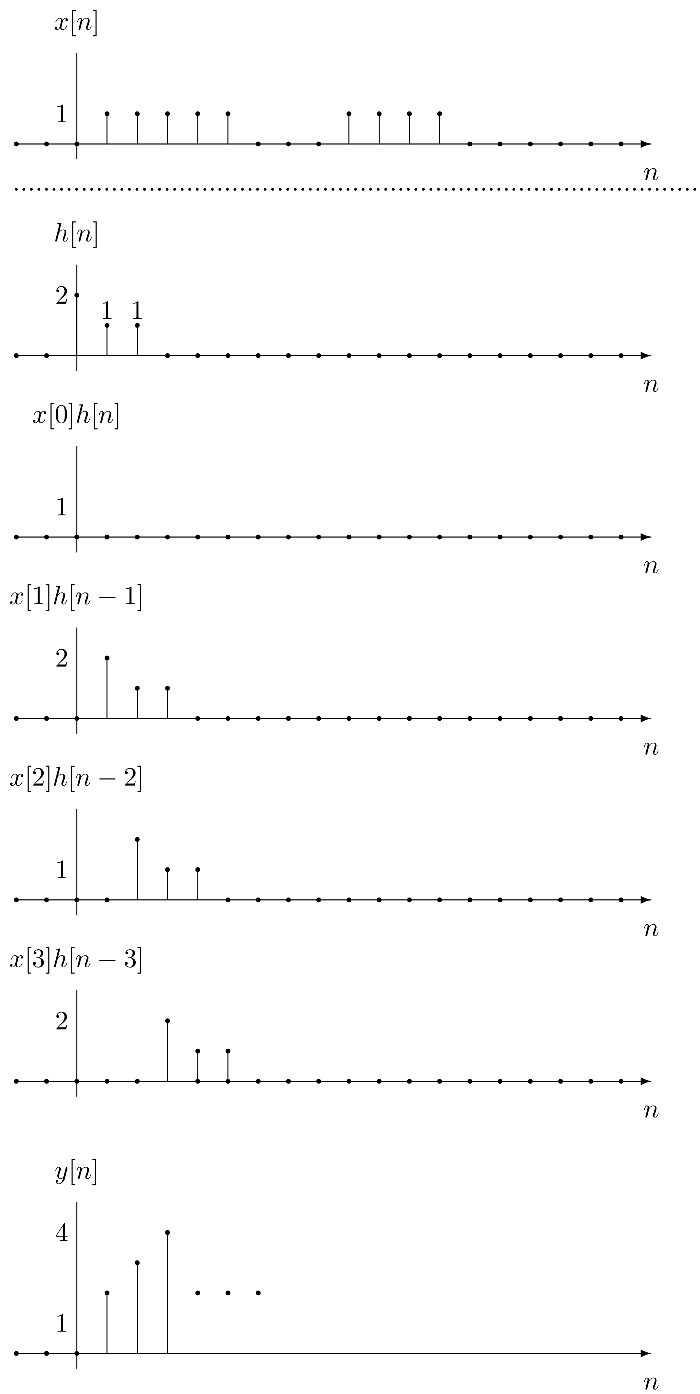
An FIR filter is given by the following difference equation:

$$y[n] = x[n-2] + x[n-1] + 2x[n].$$

We note that the impulse response $h[n]$ of the filter is given as follows.

$$h[n] = \delta[n-2] + \delta[n-1] + 2\delta[n]$$

Find the filter output $y[n]$ to the following input $x[n]$.



### 1.3 Properties of Convolution

(a) Commutative (Textbook Sec 5.6.2)

$$\begin{array}{ccc} v[n] * w[n] & = & w[n] * v[n], \\ \| & & \| \\ \sum_{k=-\infty}^{\infty} v[k]w[n-k] & & \sum_{k=-\infty}^{\infty} v[k]w[n-k] \end{array}$$

(b) Associative (Textbook Sec 5.6.2)

$$\big(v[n] * w[n]\big) * z[n] = v[n] * \big(w[n] * z[n]\big)$$

Therefore, $v[n] * w[n] * z[n]$ is unambiguous:

$$v[n] * w[n] * z[n] = \big(v[n] * w[n]\big) * z[n] = v[n] * \big(w[n] * z[n]\big)$$

(c) Distributive (not in the book)

$$\big(v[n] + w[n]\big) * z[n] = v[n] * z[n] + w[n] * z[n].$$

(d) Convolution with a finite-duration sequence (not in the book)

If $v[n] = 0$ outside of $[n_1, n_2]$, i.e., the support of $v[n]$ is $[n_1, n_2]$, then

$$\begin{aligned} v[n] * w[n] &= \sum_{k=-\infty}^{\infty} v[k]w[n-k] \\ &= \sum_{k=n_1}^{n_2} v[k]w[n-k] \\ &= \sum_{k=n-n_2}^{n-n_1} w[k]v[n-k]. \end{aligned}$$

(e) Convolution with a delta function (Textbook Sec 5.6.2)

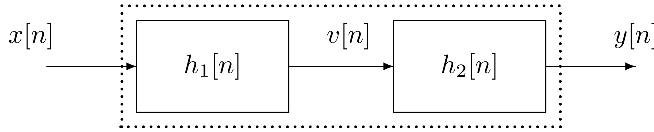
$$v[n] * \delta[n-n_0] = v[n-n_0],$$

because

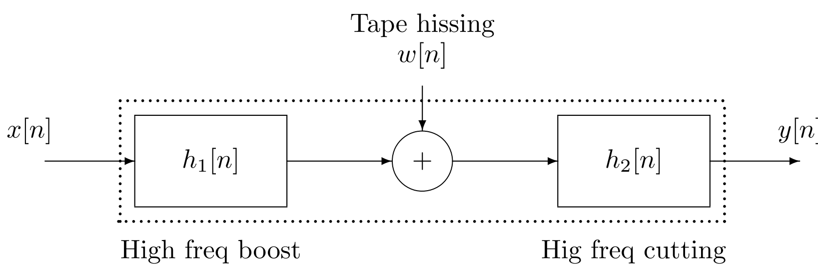
$$v[n] * \underbrace{\delta[n-n_0]}_{w[n]} = \sum_{k=-\infty}^{\infty} \underbrace{\delta[k-n_0]}_{w[k]} v[n-k] = v[n-n_0] \quad \because \delta[k-n_0] = \begin{cases} 1, & k = n_0, \\ 0, & \text{else.} \end{cases}$$

# 2 Cascaded Filters

- (Definition) Two filters are cascaded when the output of the first filter is fed to the second filter as the input:




- More than two filters can be cascaded.
- (Why Consider Cascade Connection?)
  (a) It just happens.
  The phone company filters voice signals several times in different places.
  (b) Processing
  The Dolby noise reduction system consists of two filters cascaded: one is used in recording to boost high frequency components; the other at playback to cut down boosted high frequency along with tape hissing noise.

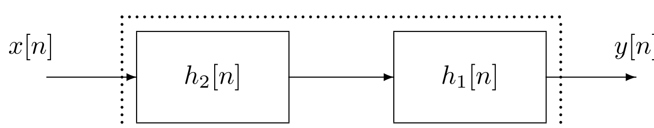



  (c) We build complex filters by cascading simple ones.
- (The Overall Impulse Response of Cascaded Filters)

$$\begin{aligned} y[n] &= v[n] * h_2[n] \\ &= \big(x[n] * h_1[n]\big) * h_2[n] \\ &= x[n] * \underbrace{\big(h_1[n] * h_2[n]\big)}_{=h[n]}. \end{aligned}$$

  (a) So, the cascade of two LTI filters is another LTI filter.
  (b) The overall impulse response is the convolution of impulse responses.
  (c) The overall impulse response does *not* depend on the order of appearance:

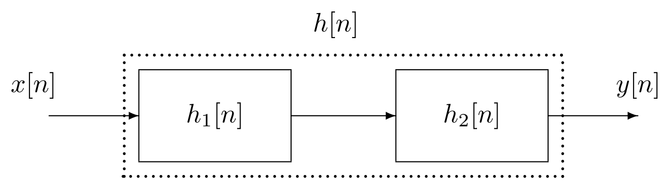
$$h_2[n] * h_1[n] = h_1[n] * h_2[n].$$

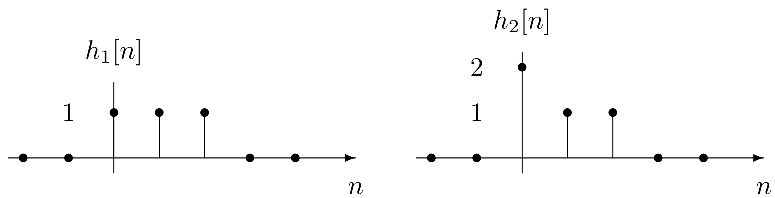



  (d) Cascaded $K$ filters $h[n] = h_1[n] * h_2[n] * \cdots h_K[n]$.

- Example

  Find the overall impulse response $h[n]$ for the following cascaded LTI filters with their respective impulse responses $h_1[n]$ and $h_2[n]$.



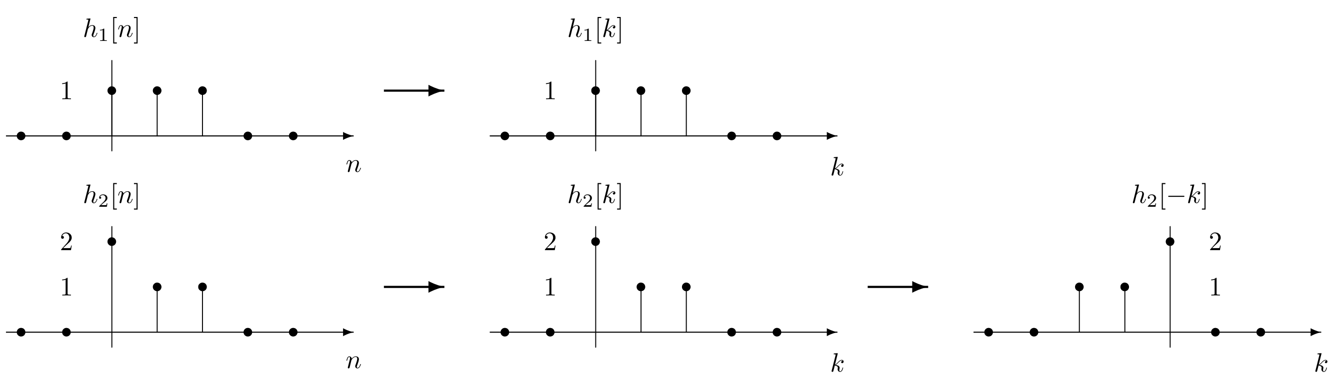




The answer is

$$h[n] = h_1[n] * h_2[n] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k].$$

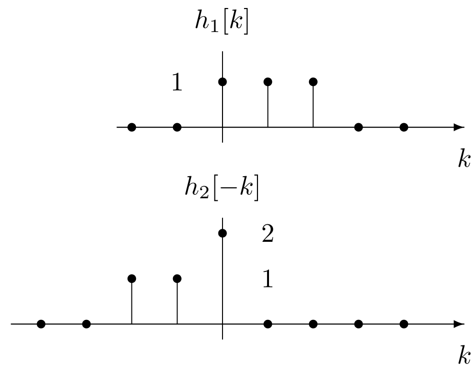
**Computational Procedure**

(a) Preparation



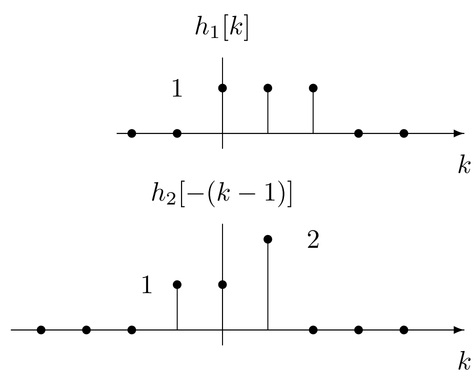

(b) $n = 0$:

$$h[n]\Big|_{n=0} = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k]\Big|_{n=0} = \sum_{k=-\infty}^{\infty} h_1[k]h_2[-k] = 2.$$



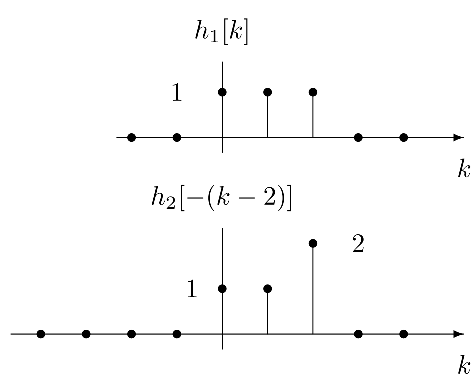

(c) $n = 1$:

$$h[n]\Big|_{n=1} = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k]\Big|_{n=1} = \sum_{k=-\infty}^{\infty} h_1[k]h_2[1-k] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[-(k-1)] = 3.$$



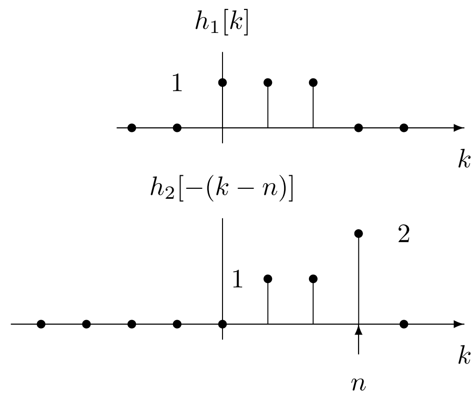

(d) $n = 2$:

$$h[n]\Big|_{n=2} = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k]\Big|_{n=2} = \sum_{k=-\infty}^{\infty} h_1[k]h_2[2-k] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[-(k-2)] = 4.$$



(e) General $n$

$$h[n] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[-(k-n)].$$




# 3 Implementation of FIR Filters

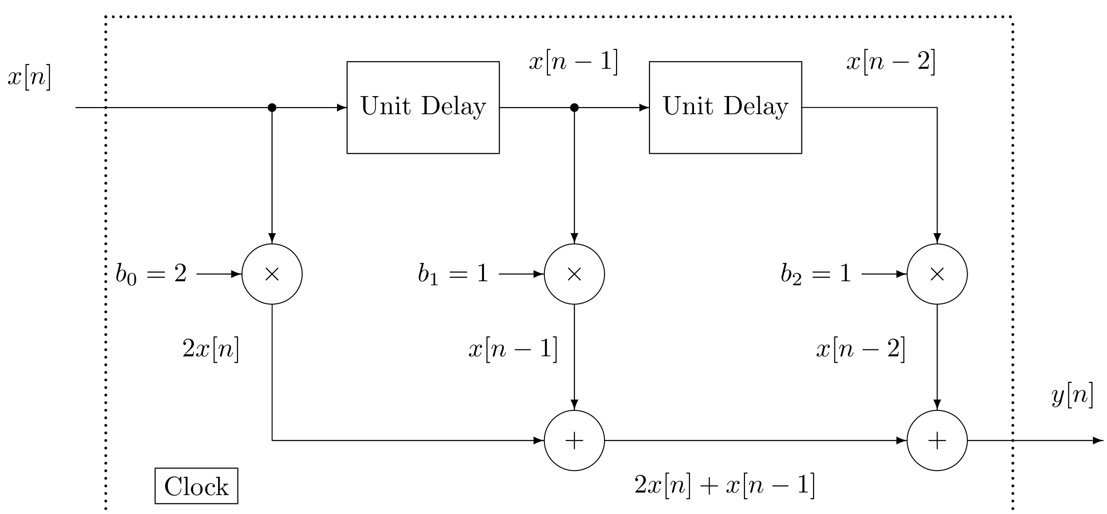
- By computer program
  (a) General purpose computer
  (b) Digital signal processor (DSP) chip
- By special purpose hardware

## Implementation by Special Purpose Hardware

Textbook: Sec 5.4

(a) Example

$$y[n] = x[n-2] + x[n-1] + 2x[n].$$



(b) Components

  (i) Multipliers
  (ii) Adders
  (iii) Unit Delays: hold the value for one unit of time and outputs the previous input value
  (iv) Conductors

(c) Operational issues

  (i) Values are passed as a bunch of bits, e.g., 8 or 16.
  (ii) Conductors are actually a set of so many parallel conductors, called the bus.
  (iii) The clock synchronizes all operations—multiplication, addition, delay, etc.

(d) Demonstration of Operation

  (i) $x[n] = \delta[n]$: then $y[n] = h[n] = \{b_0, b_1, b_2\}$
  (ii) $x[n] = \{1, 2, 2, 1, 0, 0, 0\}$

(e) Typically, there are multiple ways of implementing filters.

  (i) May use different number of components
  (ii) May have different sensitivities to finite-precision arithmetic