1 Review of LTI Filters in the Time-Domain

Table 1: Summary of LTI filter description in the time-domain

Input signal description	Filter description	Input/output relationship
$x[n]$ as a discrete-time sequence	coefficients $\{b_k\}$ of a difference equation	$y[n] = \sum_{k=-M_1}^{M_2} b_k x[n-k]$
$x[n] = x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$	impulse response $h[n]$	$y[n] = x[n] * h[n]$
	block diagram of delays, multipliers, adders	

- The filter described by $y[n] = \sum_{k=-M_1}^{M_2} b_k x[n-k]$ is LTI and has the following impulse response $h[n]$

$$h[n] = \begin{cases} b_n, & -M_1 \leq n \leq M_2, \\ 0, & \text{else.} \end{cases}$$

- Any signal $x[n]$ can be represented as

$$x[n] = x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- An LTI filter with the impulse response $h[n]$ has the response $y[n]$ to input $x[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

- Cascaded LTI filters act as one LTI filter with its overall impulse response

$$h[n] = h_1[n] * \cdots * h_K[n],$$

where $h_i[n]$ are the impulse responses of cascaded sub-filters.

2 Frequency Response of an LTI Filter

Read 6.1, 6.4, and 6.5

- A different approach to filters is possible.
  (a) Find the filter response to sinusoids/exponentials.
  (b) Express $x[n]$ as a sum of sinusoids/exponentials.
  Tools available are DFT or other spectral decompositions.
  (c) Express $y[n]$ as a sum of sinusoid/exponential responses.
- This is called the **frequency domain approach**, because we work with the frequency domain description of signals.

2.1 The Response of an LTI Filter to an Exponential Signal

- Suppose $x[n] = Ae^{j(\hat{\omega}n+\phi)} = Ae^{j\phi}e^{j\hat{\omega}n}$ is applied to a filter with impulse response $h[n]$
- Then the output $y[n]$

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]Ae^{j\phi}e^{j\hat{\omega}(n-k)} \\
&= \sum_{k=-\infty}^{\infty} h[k]Ae^{j\phi}e^{-j\hat{\omega}k}e^{j\hat{\omega}n} \\
&= \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k}\right)}_{\text{a constant determined by } h \text{ and } \hat{\omega}} \quad \underbrace{Ae^{j\phi}e^{j\hat{\omega}n}}_{x[n]} \\
&= H(\hat{\omega})x[n].
\end{aligned}$$

- Conclusion:
  (a) (Form) The output is complex exponential
  When input $x[n]$ is a *complex exponential*, the output $y[n]$ is complex exponential with the *same frequency*.
  (b) Later, we will see that

  sinusoidal input $\Longrightarrow$ sinusoid output.

  (c) (Amplitude and phase) What about the amplitude and phase of $y[n]$?

$$H(\hat{\omega}) \triangleq \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k}.$$

  This function (of $\hat{\omega}$) is called the **frequency response (function)** of the (LTI) filter/system. Then we see that

$$\begin{aligned}
y[n] &= H(\hat{\omega})\underbrace{Ae^{j\phi}e^{j\hat{\omega}n}}_{x[n]} \\
&= \big|H(\hat{\omega})\big|e^{j\angle H(\hat{\omega})}Ae^{j\phi}e^{j\hat{\omega}n} \\
&= A\big|H(\hat{\omega})\big|e^{j\left(\phi+\angle H(\hat{\omega})\right)}e^{j\hat{\omega}n}
\end{aligned}$$

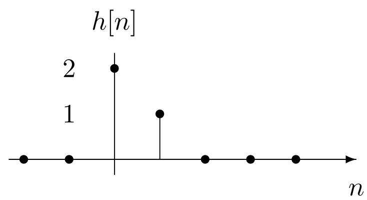
(i) The magnitude is scaled by $\left|H(\hat{\omega})\right|$.

(ii) The phase is added by $\angle H(\hat{\omega})$.

- $\left|H(\hat{\omega})\right|$ is called the *magnitude (or gain)* of the frequency response.
- $\angle H(\hat{\omega})$ is called the *phase (or angle)* of the frequency response.
- Note that both are dependent on frequency $\hat{\omega}$.

2.2 Example

An LTI filter has the following impulse response:



Find the magnitude and phase of the frequency response of the filter.

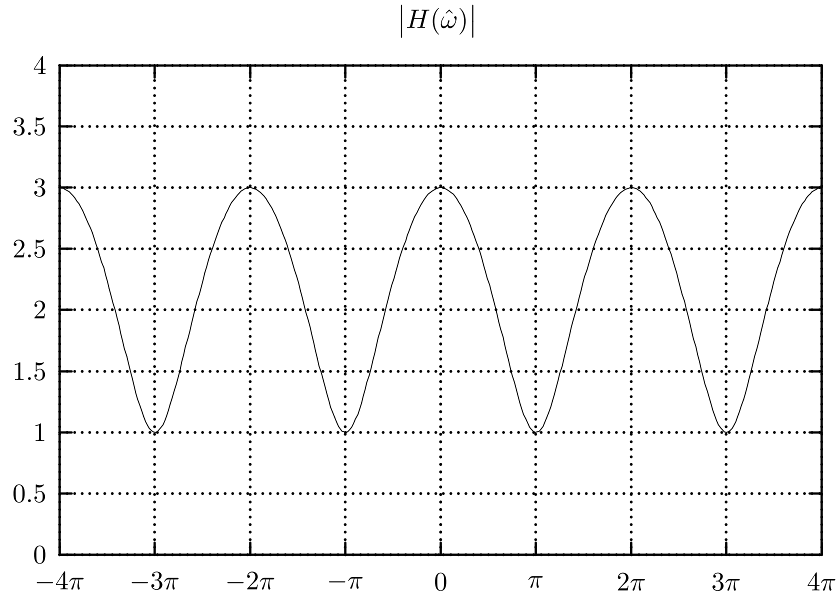
$$
\begin{aligned}
H(\hat{\omega}) &= \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k} \\
&= 2 + e^{-j\hat{\omega}} \\
&= 2 + \cos(\hat{\omega}) - j\sin(\hat{\omega}). \\
\left|H(\hat{\omega})\right| &= \sqrt{\left(2+\cos(\hat{\omega})\right)^2 + \sin^2(\hat{\omega})} \\
&= \sqrt{4 + 4\cos(\hat{\omega}) + \cos^2(\hat{\omega}) + \sin^2(\hat{\omega})} \\
&= \sqrt{5 + 4\cos(\hat{\omega})}. \\
\angle H(\hat{\omega}) &= \tan^{-1}\left(\frac{-\sin(\hat{\omega})}{2+\cos(\hat{\omega})}\right).
\end{aligned}
$$

(a) It is a lowpass filter.

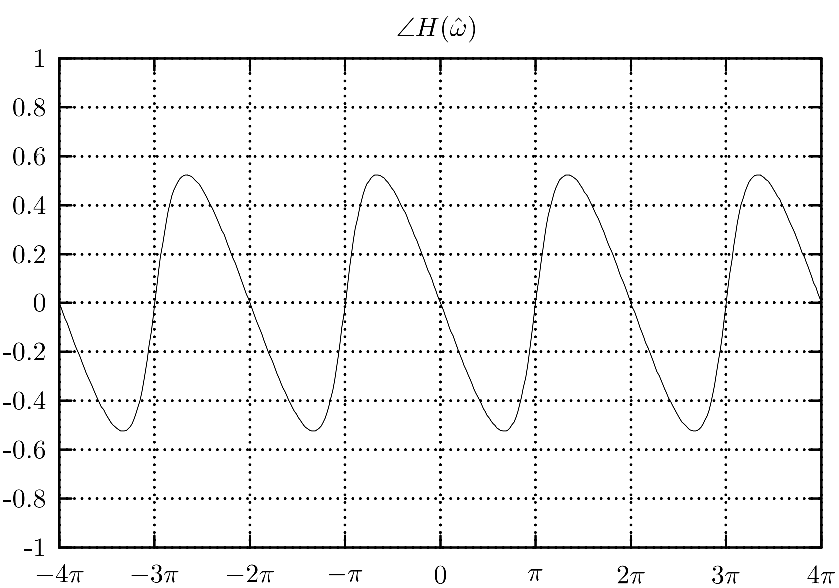
(b) Why?

Note that $\hat{\omega} = \pi$ is the highest frequency.

(c) The filter is not sharp. Note that the order of the filter is 1.



|H(ω̂)|
4
3.5
3
2.5
2
1.5
1
0.5
0
−4π
−3π
−2π
−π
0
π
2π
3π
4π



∠H(ω̂)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
−4π
−3π
−2π
−π
0
π
2π
3π
4π