1 Review: Frequency Response of an LTI Filter

- The **frequency response (function)** of an LTI filter with the impulse response $h[n]$

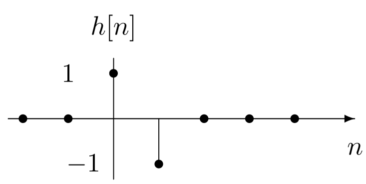
$$\mathcal{H}(\hat{\omega}) \triangleq \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k} = \big|\mathcal{H}(\hat{\omega})\big|e^{j\angle\mathcal{H}(\hat{\omega})}.$$

- Expo-in/Expo-out

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n} \quad \mapsto \quad y[n] = \mathcal{H}(\hat{\omega})x[n] = \mathcal{H}(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

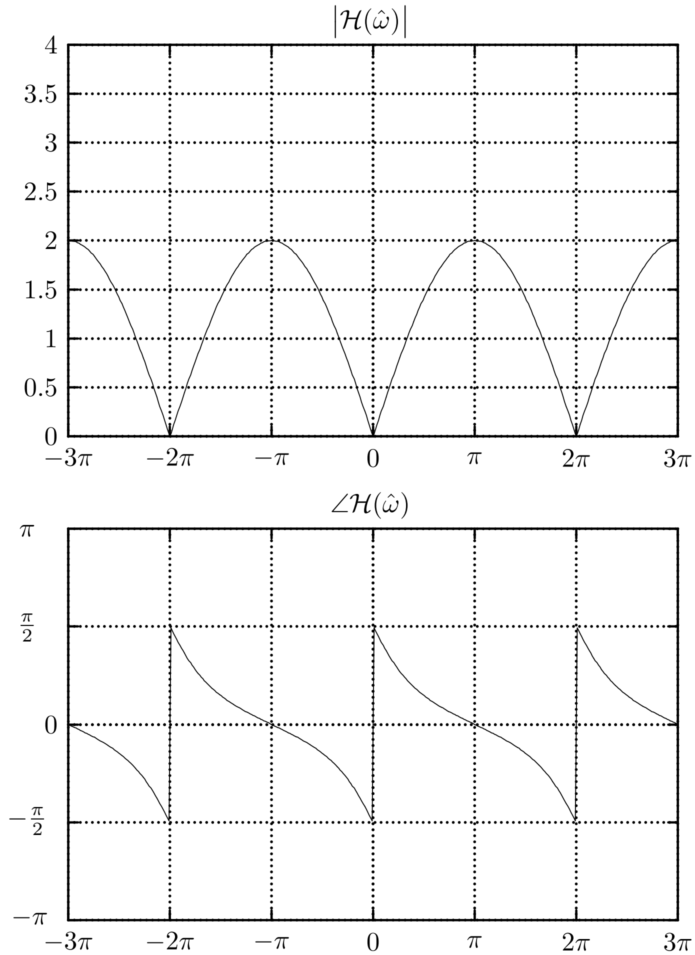
PLEDGE: I promise never to write or use $y[n] = \mathcal{H}(\hat{\omega})x[n]$ unless $x[n]$ is a complex exponential.
SIGNATURE:

- Example: first-difference filter $y[n] = x[n] - x[n-1]$.



$$\begin{aligned}
\mathcal{H}(\hat{\omega}) &= \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k} \\
&= 1 - e^{-j\hat{\omega}} \\
&= 1 - \cos(\hat{\omega}) + j\sin(\hat{\omega}). \\
\big|\mathcal{H}(\hat{\omega})\big| &= \sqrt{\big(1-\cos(\hat{\omega})\big)^2 + \sin^2(\hat{\omega})} \\
&= \sqrt{1 - 2\cos(\hat{\omega}) + \cos^2(\hat{\omega}) + \sin^2(\hat{\omega})} \\
&= \sqrt{2 - 2\cos(\hat{\omega})}. \\
\angle\mathcal{H}(\hat{\omega}) &= \tan^{-1}\left(\frac{\sin(\hat{\omega})}{2 - 2\cos(\hat{\omega})}\right) \quad \text{no need to add } \pi \because 2 - 2\cos(\hat{\omega}) \geq 0.
\end{aligned}$$

(i) Note that $\big|\mathcal{H}(0)\big| = 0$ and $\big|\mathcal{H}(\pi)\big| = 2$.
(ii) It is a highpass filter.



2 Stability

(a) Definition:

A system is said to be bounded-input/bounded-output (BIBO) **stable** if the output is bounded for a bounded input.

(b) (Boundedness) A signal $x[n]$ is bounded if there exists $M > 0$ such that

$$|x[n]| < M \quad \forall n.$$

(c) Example 1: (Stable) $y[n] = Ax[n]$.

(d) Example 2: (Unstable) $h[n] = u[n]$.

An unstable filter is a filter for which there is a bounded input that leads to an unbounded output.

Let $x[n] = u[n]$. Then $x[n] * h[n]$ is unbounded.

(e) Facts

(i) FIR filters are stable.

(ii) IIR filters can be unstable.

(iii) The frequency response $\mathcal{H}(\hat{\omega})$ is **not** defined for unstable filters

3 Properties of the Frequency Response

(a) $\mathcal{H}(\hat{\omega})$ is periodic with period 2π.

$$\mathcal{H}(\hat{\omega}+2\pi) = \mathcal{H}(\hat{\omega}).$$

(This is natural because $\hat{\omega}$ and $\hat{\omega}+2\pi n$ are equivalent.)

$$\begin{aligned}
\mathcal{H}(\hat{\omega}+2\pi) &= \sum_{k=-\infty}^{\infty} h[k]e^{-j(\hat{\omega}+2\pi)k} \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k}\underbrace{e^{-j2\pi k}}_{=1} \\
&= \mathcal{H}(\hat{\omega}).
\end{aligned}$$

(b) For a real $h[n]$, we have $\mathcal{H}(-\hat{\omega}) = \mathcal{H}^*(\hat{\omega})$.

Intuitively, it should make sense that there is nothing new in the negative frequencies. They are just there because complex exponentials are easier to work with and used as basic signals.

(i) Proof

$$\begin{aligned}
\mathcal{H}(-\hat{\omega}) &= \sum_{k=-\infty}^{\infty} h[k]e^{-j(-\hat{\omega})k} \\
&= \sum_{k=-\infty}^{\infty} h^*[k]\left(e^{-j(\hat{\omega})k}\right)^* \\
&= \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j(\hat{\omega})k}\right)^* \\
&= \mathcal{H}^*(\hat{\omega})
\end{aligned}$$

(ii) In the rectangular form

$$\begin{array}{ccc}
\mathcal{H}(-\hat{\omega}) & = & \mathcal{H}^*(\hat{\omega}) \\
\updownarrow & & \updownarrow \\
\Re\Big\{\mathcal{H}(-\hat{\omega})\Big\} + j\Im\Big\{\mathcal{H}(-\hat{\omega})\Big\} & & \Re\Big\{\mathcal{H}(\hat{\omega})\Big\} - j\Im\Big\{\mathcal{H}(\hat{\omega})\Big\}
\end{array}$$

Therefore,

$$\begin{aligned}
\Re\Big\{\mathcal{H}(-\hat{\omega})\Big\} &= \Re\Big\{\mathcal{H}(\hat{\omega})\Big\}, \\
\Im\Big\{\mathcal{H}(-\hat{\omega})\Big\} &= -\Im\Big\{\mathcal{H}(\hat{\omega})\Big\}
\end{aligned}$$

So,

$$\begin{aligned}
&\Re\Big\{\mathcal{H}(\hat{\omega})\Big\} \text{ is an even function of } \hat{\omega} \\
&\Im\Big\{\mathcal{H}(\hat{\omega})\Big\} \text{ is an odd function of } \hat{\omega}
\end{aligned}$$

(iii) In the exponential form

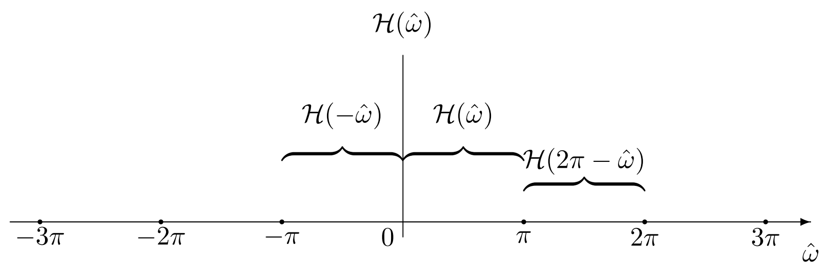
$$\begin{array}{ccc}
\mathcal{H}(-\hat{\omega}) & = & \mathcal{H}^*(\hat{\omega}) \\
\updownarrow & & \updownarrow \\
\big|\mathcal{H}(-\hat{\omega})\big|e^{j\angle\mathcal{H}(-\hat{\omega})} & & \big|\mathcal{H}(\hat{\omega})\big|e^{-j\angle\mathcal{H}(\hat{\omega})}
\end{array}$$

Equivalently,

$$\left|\mathcal{H}(-\hat{\omega})\right| = \left|\mathcal{H}(\hat{\omega})\right|,$$
$$\angle\mathcal{H}(-\hat{\omega}) = -\angle\mathcal{H}(\hat{\omega}).$$

(c) For real $h[n]$, we can restrict attention to $[0,\pi]$ because

$$\mathcal{H}(2\pi-\hat{\omega}) = \mathcal{H}(-\hat{\omega}) = \mathcal{H}^*(\hat{\omega})$$



4 The Response of an LTI Filter to a Sinusoid

(a) Statement: A sinusoid to a real LTI filter produces a sinusoid of the same frequency.

(b) Find $y[n]$ for $x[n] = A\cos(\hat{\omega}n+\phi)$.

(c) Method

(i) Decompose, using Euler's formula, the sinusoid into complex exponentials.

(ii) Use linearity of the filter.

(iii) Use CEICEO: $y[n] = \mathcal{H}(\hat{\omega})x[n]$ for $x[n] = Ae^{j(\hat{\omega}n+\phi)}$.

$$\begin{array}{lcccc}
x[n] & = & \frac{A}{2}e^{j(\hat{\omega}n+\phi)} & + & \frac{A}{2}e^{-j(\hat{\omega}n+\phi)} \\
\downarrow & & & & \\
y[n] & = & \mathcal{H}(\hat{\omega})\frac{A}{2}e^{j(\hat{\omega}n+\phi)} & + & \mathcal{H}(-\hat{\omega})\frac{A}{2}e^{-j(\hat{\omega}n+\phi)} \\
 & = & \left|\mathcal{H}(\hat{\omega})\right|e^{j\angle\mathcal{H}(\hat{\omega})}\frac{A}{2}e^{j(\hat{\omega}n+\phi)} & + & \left|\mathcal{H}(-\hat{\omega})\right|e^{j\angle\mathcal{H}(-\hat{\omega})}\frac{A}{2}e^{-j(\hat{\omega}n+\phi)} \\
 & = & \left|\mathcal{H}(\hat{\omega})\right|e^{j\angle\mathcal{H}(\hat{\omega})}\frac{A}{2}e^{j(\hat{\omega}n+\phi)} & + & \left|\mathcal{H}(\hat{\omega})\right|e^{-j\angle\mathcal{H}(\hat{\omega})}\frac{A}{2}e^{-j(\hat{\omega}n+\phi)} \\
 & = & \left|\mathcal{H}(\hat{\omega})\right|\frac{A}{2}e^{j\left(\hat{\omega}n+\phi+\angle\mathcal{H}(\hat{\omega})\right)} & + & \left|\mathcal{H}(\hat{\omega})\right|\frac{A}{2}e^{-j\left(\hat{\omega}n+\phi+\angle\mathcal{H}(\hat{\omega})\right)},
\end{array}$$

Therefore,

$$y[n] = \left|\mathcal{H}(\hat{\omega})\right|A\cos\left(\hat{\omega}n+\phi+\angle\mathcal{H}(\hat{\omega})\right).$$

(d) Conclusion:

For a sinusoidal input with frequency $\hat{\omega}$, the response of a real LTI filter is a sinusoid with the same frequency, with amplitude scaled by $\left|\mathcal{H}(\hat{\omega})\right|$ and phase added by $\angle\mathcal{H}(\hat{\omega})$.

4.1 Example of a Sinusoidal Response

Let

$$h[n] = \begin{cases} 2, & n = 0, \\ 1, & n = 1, \\ 0, & \text{else.} \end{cases}$$

Find the response of the filter to

$$x[n] = 4 + 3\cos(n + 0.1) + 4\cos(\sqrt{2}n + 0.2).$$

Solution:

(a) Note that the input is a weighted sum of sinusoids.

Linearity of the filter implies that the output will be a weighted sum of individual sinusoidal responses.

$$x[n] = \underbrace{4}_{x_1[n]} + \underbrace{3\cos(n + 0.1)}_{x_2[n]} + \underbrace{4\cos(\sqrt{2}n + 0.2)}_{x_3[n]}.$$

(b) Individual sinusoidal responses given $\mathcal{H}(\hat{\omega})$

$$\begin{aligned} y_1[n] &= \mathcal{H}(0)4\cos\big(0 + \angle\mathcal{H}(0)\big), \\ y_2[n] &= \mathcal{H}(1)3\cos\big(n + 0.1 + \angle\mathcal{H}(1)\big), \\ y_1[n] &= \mathcal{H}(\sqrt{2})4\cos\big(\sqrt{2}n + 0.2 + \angle\mathcal{H}(\sqrt{2})\big). \end{aligned}$$

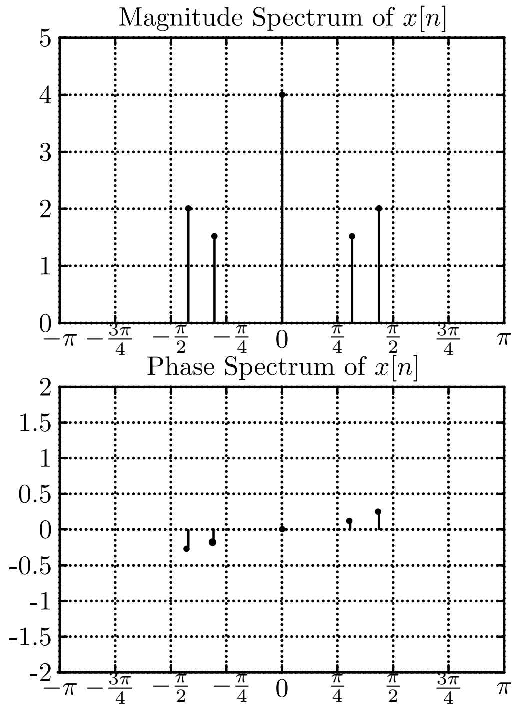
Therefore

$$\begin{aligned} y[n] &= 3 \cdot 4 + 2.68 \cdot 3\cos\big(n + 0.1 - 0.32\big) + 2.37 \cdot 4\cos\big(\sqrt{2}n + 0.2 - 0.43\big) \\ &= 12 + 8.04\cos\big(n - 0.22\big) + 9.48\cos\big(\sqrt{2}n - 0.23\big). \end{aligned}$$

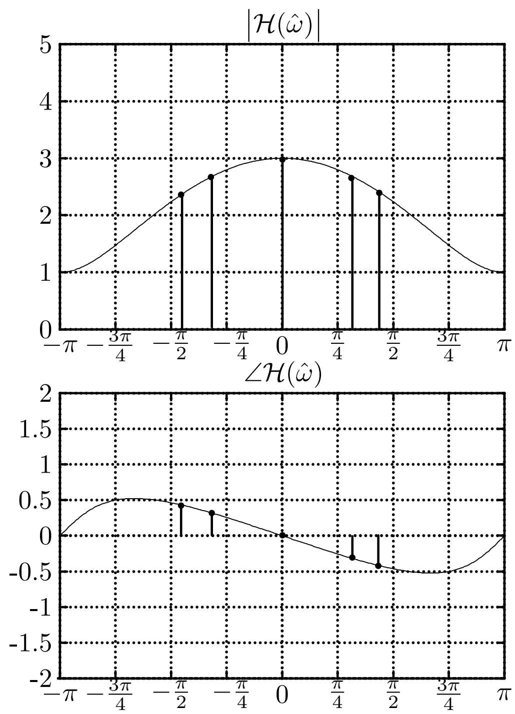
The frequency response $\mathcal{H}(\hat{\omega})$:

$$\begin{aligned} \mathcal{H}(\hat{\omega}) &= 2 + e^{-j\hat{\omega}} \\ &= 2 + \cos(\hat{\omega}) - j\sin(\hat{\omega}). \\ \big|\mathcal{H}(\hat{\omega})\big| &= \sqrt{5 + 4\cos(\hat{\omega})}. \\ \angle\mathcal{H}(\hat{\omega}) &= \tan^{-1}\Big(\frac{-\sin(\hat{\omega})}{2 + \cos(\hat{\omega})}\Big). \end{aligned}$$

(c) Visualization in the frequency domain.



Magnitude Spectrum of $x[n]$
Phase Spectrum of $x[n]$
$-\pi$ $-\frac{3\pi}{4}$ $-\frac{\pi}{2}$ $-\frac{\pi}{4}$ 0 $\frac{\pi}{4}$ $\frac{\pi}{2}$ $\frac{3\pi}{4}$ π



$\left|\mathcal{H}(\hat{\omega})\right|$
$\angle\mathcal{H}(\hat{\omega})$
$-\pi$ $-\frac{3\pi}{4}$ $-\frac{\pi}{2}$ $-\frac{\pi}{4}$ 0 $\frac{\pi}{4}$ $\frac{\pi}{2}$ $\frac{3\pi}{4}$ π