1 Review and Topics

The frequency response of an LTI filter with the impulse response $h[n]$ is

$$\mathcal{H}(\hat{\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\hat{\omega}k}.$$

- The response to a complex exponential is complex exponential

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n} \quad \mapsto \quad y[n] = \mathcal{H}(\hat{\omega})x[n] = \mathcal{H}(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

  Sum of complex exponentials:

$$x[n] = \sum_l A_l e^{j\phi_l} e^{j\hat{\omega}_l n} \quad \mapsto \quad y[n] = \sum_l \mathcal{H}(\hat{\omega}_l) A_l e^{j\phi_l} e^{j\hat{\omega}_l n}$$

- The response to a sinusoid

$$x[n] = A\cos(\hat{\omega}n + \phi) \quad \mapsto \quad y[n] = A\big|\mathcal{H}(\hat{\omega})\big| \cos\big(\hat{\omega}n + \phi + \angle\mathcal{H}(\hat{\omega})\big).$$

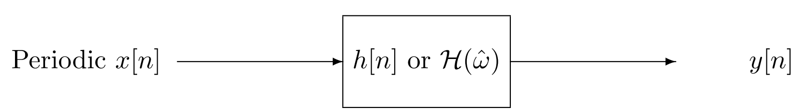
  Sum of sinusoids:

$$x[n] = \sum_l A_l \cos(\hat{\omega}_l n + \phi_l) \quad \mapsto \quad y[n] = \sum_l A_l \big|\mathcal{H}(\hat{\omega}_l)\big| \cos\big(\hat{\omega}_l n + \phi_l + \angle\mathcal{H}(\hat{\omega}_l)\big)$$

- The response to a periodic signal
- The response to a suddenly-applied signal

2 Periodic Signals and LTI Systems

2.1 The Response of an LTI Filter to a Periodic Signal



(a) A periodic signal with period N_0 can be decomposed into a sum of complex exponential

$$x[n] = \sum_{k=0}^{N_0-1} X[k] e^{j\frac{2\pi k}{N_0}n},$$

where

$$X[k] = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi k}{N_0}n}, \quad k = 0, \ldots, N_0 - 1.$$

(b) Then for each component signal

$$X[k]e^{j\frac{2\pi k}{N_0}n} \quad \mapsto \quad X[k]\mathcal{H}\Big(\frac{2\pi k}{N_0}\Big)e^{j\frac{2\pi k}{N_0}n}$$

(c) Therefore, the linearity of the filter yields

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_0-1} X[k]\mathcal{H}\Big(\frac{2\pi k}{N_0}\Big)e^{j\frac{2\pi k}{N_0}n} \\ &= \sum_{k=0}^{N_0-1} X[k]\Big|\mathcal{H}\Big(\frac{2\pi k}{N_0}\Big)\Big|e^{j\left(\frac{2\pi k}{N_0}n+\angle\mathcal{H}\left(\frac{2\pi k}{N_0}\right)\right)}. \end{aligned}$$

(d) Comments

(i) This is not the sort that is usually processed by hand.

(ii) But it is a powerful tool.

(iii) An aperiodic signal can be dealt with using this technique by dividing it into blocks.

(e) DFT Fact:

(i) When $x[n]$ is periodic with N_0, so is $y[n]$.

(ii) So

$$y[n] = \sum_{k=0}^{N_0-1} Y[k]e^{j\frac{2\pi k}{N_0}n},$$

where $Y[k]$ $(k = 0, \ldots, N_0 - 1)$ are the N_0-point DFT coefficients of $y[n]$.

(iii) But we know that

$$y[n] = \sum_{k=0}^{N_0-1} \underbrace{X[k]\mathcal{H}\Big(\frac{2\pi k}{N_0}\Big)}_{=Y[k]} e^{j\frac{2\pi k}{N_0}n},$$

because DFT is one-to-one (i.e., there is only one set of DFT coefficients for a signal).

(iv) Therefore, the N_0-point DFT coefficients of $y[n]$ are given by

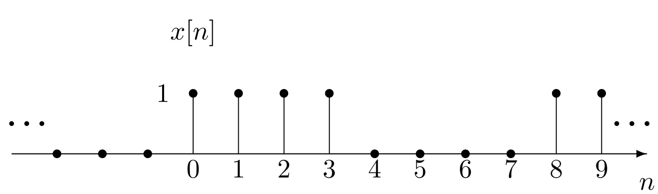
$$Y[k] = X[k]\mathcal{H}\Big(\frac{2\pi k}{N_0}\Big).$$

2.2 Example of a Periodic Response

Let

$$h[n] = \begin{cases} 2, & n = 0, \\ -2, & n = 1, \\ 0, & \text{else.} \end{cases}$$

Find the response of the filter to the following $x[n]$:



Solution:

(a) The LTI filter has the following frequency response

$$\begin{aligned}
\mathcal{H}(\hat{\omega}) &= \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k} \\
&= 2 - 2e^{-j\hat{\omega}}. \\
\left|\mathcal{H}(\hat{\omega})\right| &= \sqrt{8 - 8\cos(\hat{\omega})}. \\
\angle\mathcal{H}(\hat{\omega}) &= \tan^{-1}\Big(\frac{\sin(\hat{\omega})}{1-\cos(\hat{\omega})}\Big).
\end{aligned}$$

(b) Note that the input is periodic with period $N_0 = 8$.

$$x[n] = \sum_{k=0}^{7} X[k]e^{j\frac{2\pi k}{8}n},$$

where

$$X[k] = \frac{1}{8}\sum_{n=0}^{7} x[n]e^{-j\frac{2\pi k}{N_0}n}, \quad k = 0, \ldots, N_0 - 1.$$

And the response $y[n]$ is

$$\begin{aligned}
y[n] &= \sum_{k=0}^{7} X[k]\mathcal{H}\Big(\frac{2\pi k}{8}\Big)e^{j\frac{2\pi k}{8}n} \\
&= \sum_{k=0}^{7} X[k]\big|\mathcal{H}\Big(\frac{2\pi k}{8}\Big)\big|e^{j\Big(\frac{2\pi k}{8}n + \angle\mathcal{H}\big(\frac{2\pi k}{8}\big)\Big)}.
\end{aligned}$$

(c) Find the 8-point DFT coefficients $X[k]$ for $k = 0, \ldots, N_0 - 1$.

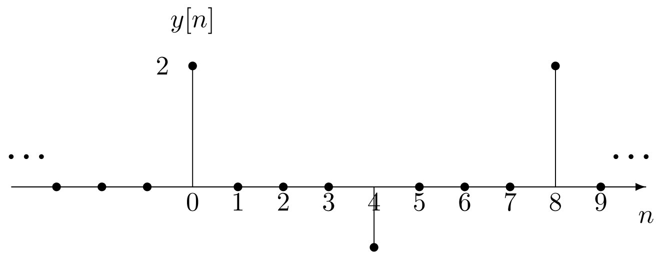
$$\begin{aligned}
X[k] &= \frac{1}{8}\sum_{n=0}^{4} e^{-j\frac{2\pi k}{8}n} = \frac{1}{8}\sum_{n=0}^{4}\underbrace{\big(e^{-j\frac{2\pi k}{8}}\big)^n}_{\alpha^n} \\
&= \begin{cases} \frac{1}{8}\frac{1-\alpha^4}{1-\alpha}, & \alpha \neq 1, \\ \frac{1}{2}, & \alpha = 0, \end{cases} \\
&= \begin{cases} \frac{1}{2}, & k = 0, \\ 0, & k = 2, 4, 6, \\ \frac{1}{4}\frac{1}{1-e^{-j\pi k/4}}, & k = 1, 3, 5, 7. \end{cases}
\end{aligned}$$

(d) Then $Y[k]$ from $\mathcal{H}\Big(\frac{2\pi k}{8}\Big) = 2 - 2e^{-j2\pi k/8}$

$$\begin{aligned}
Y[k] = X[k]\mathcal{H}\Big(\frac{2\pi k}{8}\Big) &= \begin{cases} \frac{1}{2}\mathcal{H}\Big(\frac{2\pi k}{8}\Big), & k = 0, \\ 0 \cdot \mathcal{H}\Big(\frac{2\pi k}{8}\Big), & k = 2, 4, 6, \\ \frac{1}{4}\frac{1}{1-e^{-j\pi k/4}}\mathcal{H}\Big(\frac{2\pi k}{8}\Big), & k = 1, 3, 5, 7. \end{cases} \\
&= \begin{cases} 0, & k = 0, 2, 4, 6, \\ \frac{1}{2}, & k = 1, 3, 5, 7. \end{cases}
\end{aligned}$$

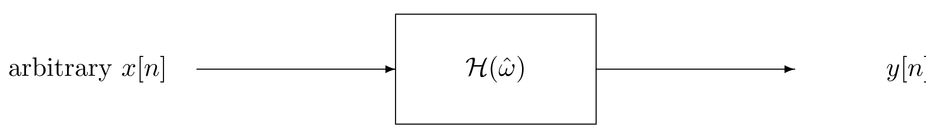
(e) Then the response $y[n]$

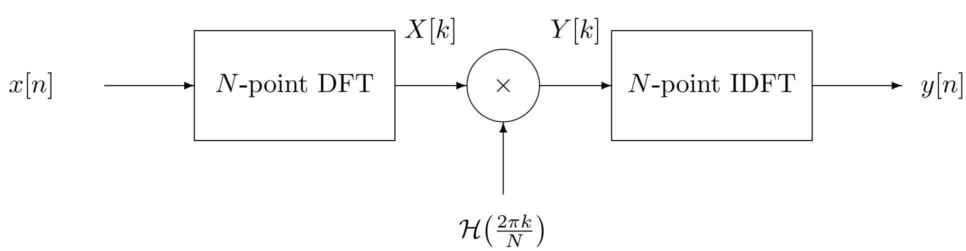
$$
\begin{aligned}
y[n] &= \sum_{k=0}^{7} Y[k] e^{j\frac{2\pi k}{8} n} \\
&= \underbrace{\frac{1}{2} e^{j2\pi n/8}}_{a} + \underbrace{\frac{1}{2} e^{j2\pi 3n/8}}_{b} + \underbrace{\frac{1}{2} e^{j2\pi 5n/8}}_{c} + \underbrace{\frac{1}{2} e^{j2\pi 7n/8}}_{d} \\
&= (a+d) + (b+c) \\
&= \frac{1}{2}\left(e^{j2\pi n/8} + e^{-j2\pi n/8}\right) + \frac{1}{2}\left(e^{j2\pi 3n/8} + e^{-j2\pi 3n/8}\right) \\
&= \cos\left(2\pi n/8\right) + \cos\left(2\pi 3n/8\right).
\end{aligned}
$$

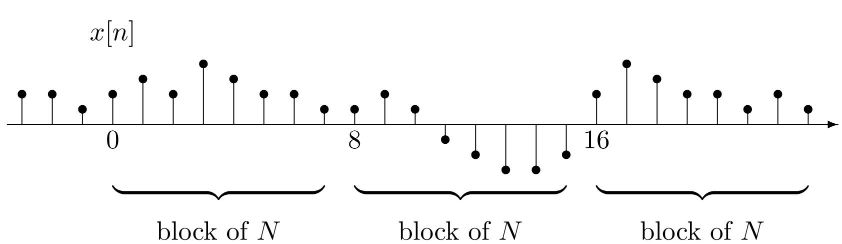


2.3 Implementation of Filtering via DFT

Sometime filtering is implemented via DFT.







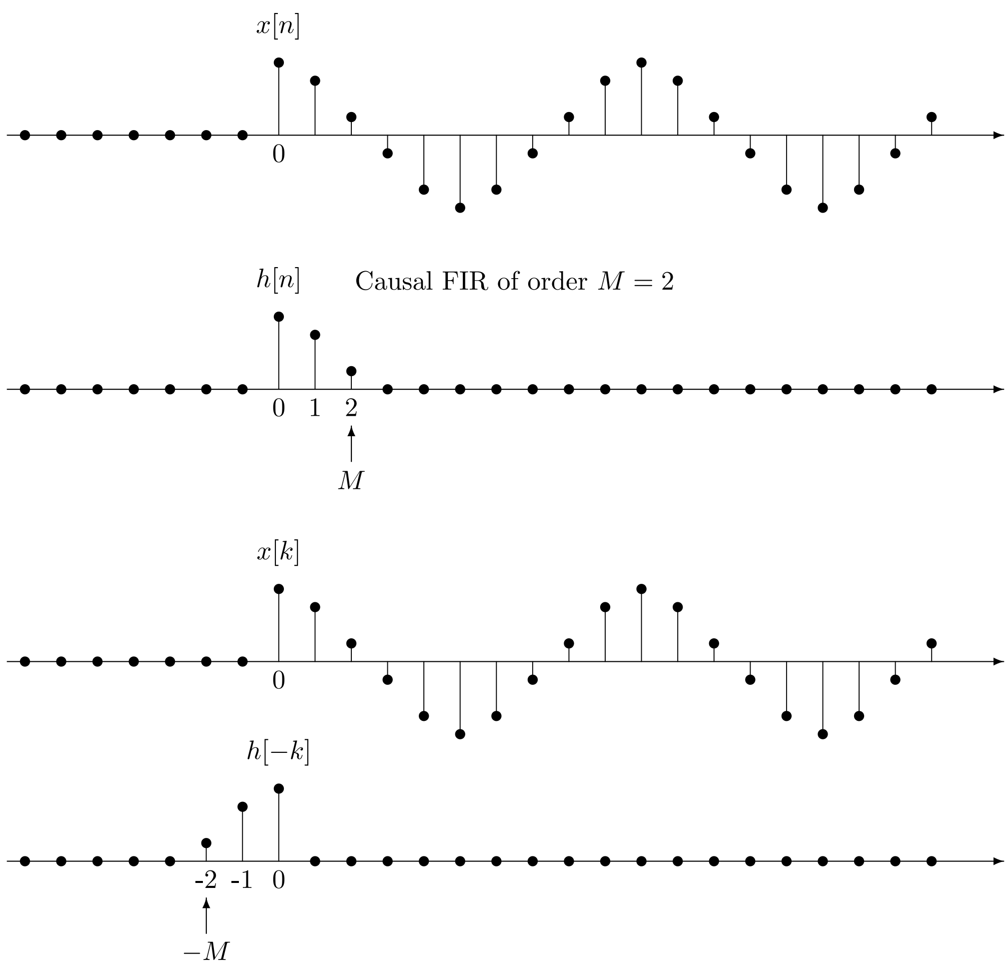
- Choose N, e.g., $N = 256$.
- Divide the signal into blocks of N samples.
- Take the N-point DFT of the block.
- Multiply by $\mathcal{H}\left(\frac{2\pi k}{N}\right)$
- Obtain the DFT coefficients of the output block.
- Take the N-point IDFT.
- Repeat for the next block.

3 Suddenly-Applied Signals

(a) When a signal is suddenly applied, it takes time for the output to reach its normal operating condition.

  (i) Steady-state region: normal operation is established.

  (ii) Transient region: the output builds up to the steady state.



(b) Example

$$x[n] = \begin{cases} Ae^{j\phi}e^{j\hat{\omega}n}, & n \geq 0, \\ 0, & n < 0, \end{cases}$$
$$= Ae^{j\phi}e^{j\hat{\omega}n}u[n],$$

where $u[n]$ is the unit-step signal

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases}$$

(c) How to find the response of an LTI filter to a suddenly-applied signal.

Approaches:

(i) the time-domain: $y[n] = x[n] * h[n]$.

(ii) the frequency-domain? Does not entirely work, because $x[n]$ is not complex exponential, sinusoidal, a sum of such, or periodic.

(iii) Special case of a *causal* LTI filter of order M:

$$y[n] = \sum_{k=0}^{M} h[k]Ae^{j\phi}e^{j\hat{\omega}(n-k)}u[n-k]$$
$$= \begin{cases} 0, & n < 0, \\ \sum_{k=0}^{n} h[k]Ae^{j\phi}e^{j\hat{\omega}(n-k)}, & 0 \leq n \leq M-1, \\ \mathcal{H}(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}, & n \geq M. \end{cases}$$

Three regions:

(1) $n < 0$: zero output due to causality

(2) $0 \leq n \leq M - 1$: transient region—use the time-domain approach

(3) $n \geq M$: steady-state region—use the frequency-domain approach

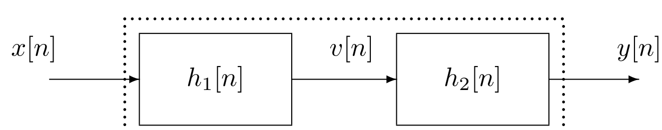
(d) Extensions are possible to other kinds of input, such as sums of complex exponentials, sinusoids, periodic signals.

See Example 6.5 on page 165 of the text for a suddenly-applied sinusoid.

4 Cascaded Filters in the Frequency Domain

4.1 Review: Cascaded Filters in the Time Domain

- Two filters cascaded:

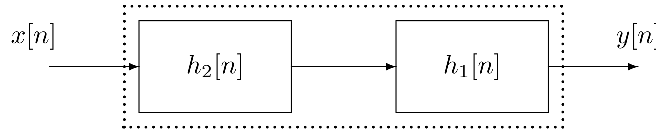


- The Overall Impulse Response of Cascaded Filters: $h[n]$

$$h[n] = h_1[n] * h_2[n].$$

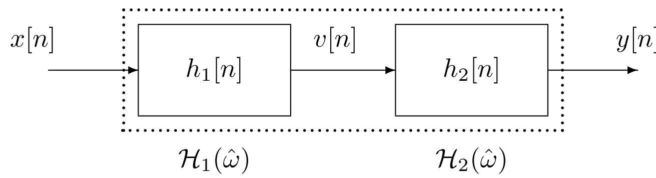
- The overall impulse response does *not* depend on the order of appearance:

$$h[n] = h_2[n] * h_1[n] = h_1[n] * h_2[n].$$



4.2 Frequency-Domain Description

- The frequency response of two cascaded LTI filters is the product of individual frequency responses.



$$\begin{aligned} h[n] &= h_1[n] * h_2[n], \\ \mathcal{H}(\hat{\omega}) &= \mathcal{H}_1(\hat{\omega})\mathcal{H}_2(\hat{\omega}). \end{aligned}$$

- Derivation

$$\begin{aligned} x[n] = \underbrace{Ae^{j\phi}}_{X} e^{j\hat{\omega}n} &\quad \mapsto v[n] = \mathcal{H}_1(\hat{\omega}) X e^{j\hat{\omega}n} = \mathcal{H}_1(\hat{\omega}) A e^{j\phi} e^{j\hat{\omega}n}, \\ v[n] = \underbrace{\mathcal{H}_1(\hat{\omega}) Ae^{j\phi}}_{V} e^{j\hat{\omega}n} &\quad \mapsto \quad y[n] = \mathcal{H}_2(\hat{\omega}) \cdot V e^{j\hat{\omega}n} = \mathcal{H}_2(\hat{\omega})\mathcal{H}_1(\hat{\omega}) A e^{j\phi} e^{j\hat{\omega}n} \\ &\quad = \underbrace{\mathcal{H}_2(\hat{\omega})\mathcal{H}_1(\hat{\omega})}_{\mathcal{H}(\hat{\omega})} x[n]. \end{aligned}$$

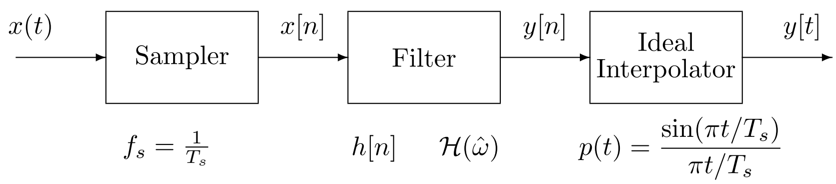
- Order of filters does not matter.
- In general, convolution in the time domain corresponds to multiplication in the frequency domain.

$$h[n] = h1[n] * h_2[n] \quad \Longleftrightarrow \quad \mathcal{H}(\hat{\omega}) = \mathcal{H}_1(\hat{\omega})\mathcal{H}_2(\hat{\omega}).$$

$$\begin{aligned} \mathcal{H}(\hat{\omega}) &= \sum_{k=-\infty}^{\infty} h[k] e^{-jk\hat{\omega}} \\ &= \sum_{k=-\infty}^{\infty} \Big(\sum_{l=-\infty}^{\infty} h_1[l] h_2[k-l] \Big) e^{-jk\hat{\omega}} \\ &= \sum_{l=-\infty}^{\infty} h_1[l] \sum_{k=-\infty}^{\infty} h_2[k-l] e^{-jk\hat{\omega}} \\ &= \sum_{l=-\infty}^{\infty} h_1[l] \Big(\sum_{m=-\infty}^{\infty} h_2[m] e^{-jm\hat{\omega}} \Big) e^{-jl\hat{\omega}} \qquad \because m = k - l \\ &= \mathcal{H}_1(\hat{\omega})\mathcal{H}_2(\hat{\omega}). \end{aligned}$$

5 Filtering of Continuous-Time Signals Using Sampling and Discrete-Time Filters

(a) Structure



(b) Defining Characteristics: f_s or T_s; $h[n]$ or $\mathcal{H}_1(\hat{\omega})$.

$$\begin{aligned} x[n] &= x(nT_s), \\ y[n] &= x[n] * h[n], \\ y(t) &= \sum_n y[n]p(t - nT_s), \quad p(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s}. \end{aligned}$$

We note that the system is linear but not time-invariant. But we just need linearity here.

(c) Example $x(t) = Xe^{j\omega_0 t}$.

(i) Choose $f_s > 2f_{\max} = 2\frac{\omega_0}{2\pi}$ or $\omega_0 T_s < \pi$ for perfect reconstruction.

(ii) Then $x[n] = Xe^{j\omega_0 nT_s} = Xe^{j\hat{\omega}_0 n}$, $\hat{\omega}_0 = \omega_0 T_s$.

(iii) Then CICO implies that

$$y[n] = \mathcal{H}(\hat{\omega}_0)Xe^{j\hat{\omega}_0 n}.$$

(iv) Find $y(t)$.
Answer:

$$\begin{aligned} y(t) &= X\mathcal{H}(\hat{\omega}_0)e^{j\omega_0 t} \\ &= X\mathcal{H}(\omega_0 T_s)e^{j\omega_0 t}, \end{aligned}$$

because $y[n]$ has frequencies $\hat{\omega} < \pi$.
In conclusion

$$x(t) = Xe^{j\omega_0 t} \quad \mapsto \quad y(t) = \mathcal{H}(\omega_0 T_s)Xe^{j\omega_0 t} = \mathcal{H}(\omega_0 T_s)x(t).$$

(v) This result is applicable to sums of complex exponentials, sinusoids, sums of sinusoids and periodic signals.