The System Function of IIR Filters

An N-th order IIR filter described by a difference equation

$$\begin{aligned} y[n] =& a_1 y[n-1] + a_2 y[n-2] + \cdots + a_N y[n-N] \\ &+ b_0 x[n] + b_1 x[n-2] + \cdots + b_M x[n-M] \end{aligned}$$

has the system function $H(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 - a_1 z^{-1} - \cdots - a_N z^{-N}}.$$

(1) **The frequency response function** $\mathcal{H}(\hat{\omega})$ is found by

$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}).$$

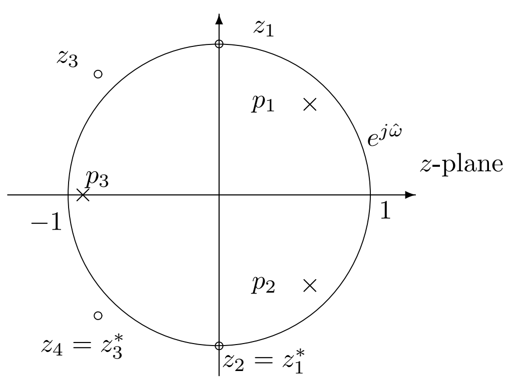
Only stable filters have $\mathcal{H}(\hat{\omega})$.

(2) **Poles and zeros**

$$H(z) = G\frac{(z-z_1)(z-z_2)\cdots(z-z_K)}{(z-p_1)(z-p_2)\cdots(z-p_L)},$$

where

$$\begin{aligned} \text{zeros}: &\quad z_1, z_2, \ldots, z_K, \\ \text{poles}: &\quad p_1, p_2, \ldots, z_L. \end{aligned}$$

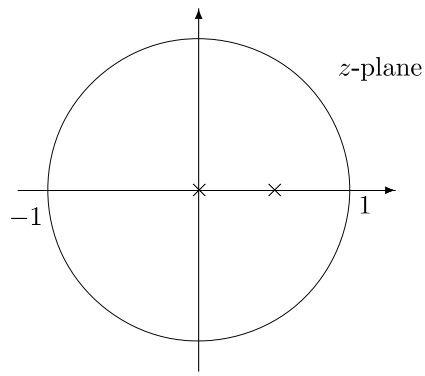


(i) The system function is uniquely determined by the gain, zeros and poles.

(ii) Poles and zeros come in conjugate pairs (for real-coefficient filters).

(iii) There are always an equal number of poles and zeros when poles and zeros at the origin and infinity are included.
Example of zeros at ∞: $y[n] = 0.5y[n-1] + 2x[n-2]$. Note that

$$H(z) = \frac{2z^{-2}}{1-0.5z^{-1}} = \frac{2}{z(z-0.5)}.$$



Two poles are at $z = 0$ and $z = 0.5$ and has two zeros at ∞:

$$
\begin{aligned}
H(z) &\to 0 \quad z \to \infty, \\
zH(z) &\to 0 \quad z \to \infty.
\end{aligned}
$$

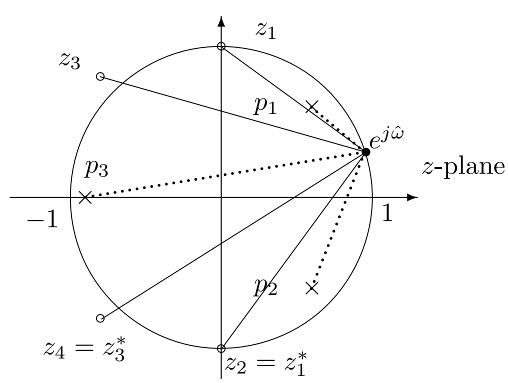
(iv) **Stability:** Locations of poles determine stability. (Details later)

Stable IIR filters $\Longleftrightarrow$ all poles inside the unit circle.

(v) **Estimation of the shape of $|\mathcal{H}(\hat{\omega})|$**

The poles and zeros provide an extremely useful characterization of the filter. (The poles and zeros will give us a good idea of the magnitude frequency response).

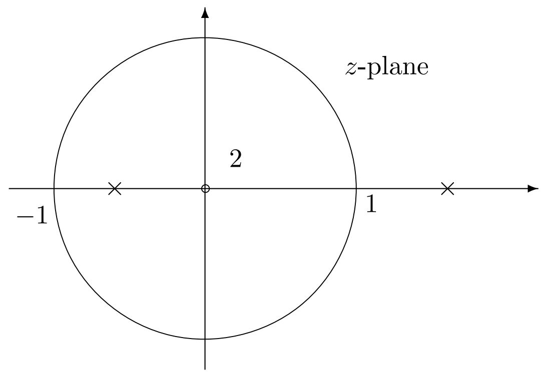
$$
\begin{aligned}
|\mathcal{H}(\hat{\omega})| = &= |H(e^{j\hat{\omega}})| \\
&= |G| \frac{|e^{j\hat{\omega}} - z_1| \cdot |e^{j\hat{\omega}} - z_2| \cdots |e^{j\hat{\omega}} - z_K|}{|e^{j\hat{\omega}} - p_1| \cdot |e^{j\hat{\omega}} - p_2| \cdots |e^{j\hat{\omega}} - p_L|} \\
&= |G| \frac{\prod_i^K d(e^{j\hat{\omega}}, z_i)}{\prod_i^L d(e^{j\hat{\omega}}, p_i)}.
\end{aligned}
$$



- poles: peaks
- zeros: nulls or dips

(a) Example:

$$H(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{z^2}{z^2 - z - 1}$$
$$= \frac{z^2}{(z + 0.6180)(z - 1.6180)}$$

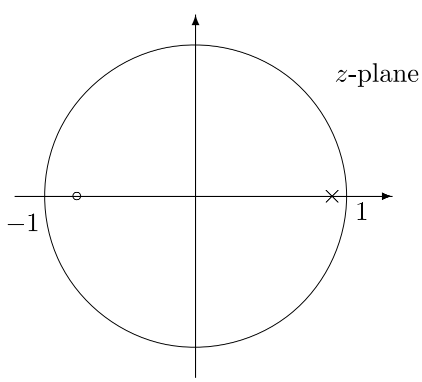


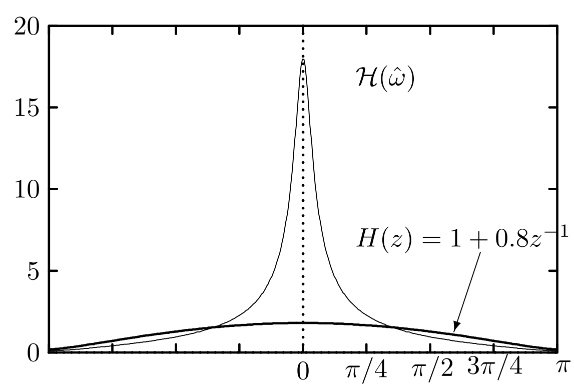
The filter is unstable and hence $\mathcal{H}(\hat{\omega})$ does not exist.

(b) Example:

$$H(z) = \frac{1 + 0.8z^{-1}}{1 - 0.9z^{-1}}$$
$$= \frac{z + 0.8}{z - 0.9}$$

The filter is stable and $\mathcal{H}(\hat{\omega})$ has a peak at $\hat{\omega} = 0$ and a dip at $\hat{\omega} = \pi$.

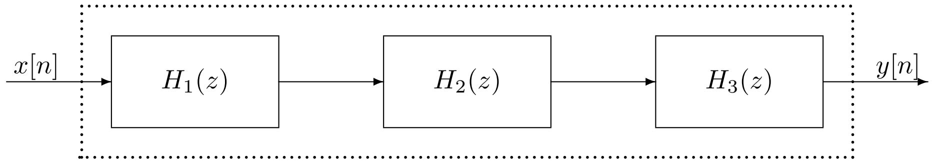




(3) **IIR vs FIR**

IIR filters are sharper due to poles than FIR filters.

(4) **Cascaded Filters** The overall system function $H(z)$ of cascaded filters is the product of component system functions.



$$H(z) = H_1(z)H_2(z)H_3(z)$$
$$y[n] = h_1[n] * h_2[n] * h_3[n] * x[n]$$

Usefulness A complicated filter is often designed as cascaded filters of order 1 and order 2 as building blocks.

Example: Given

$$\begin{aligned}
H(z) &= \frac{1 + z^{-1} + z^{-2} + z^{-3}}{1 - 1.556z^{-1} + 1.272z^{-2} - 0.398z^{-3}} \\
&= \frac{(1 + z^{-1})(1 - e^{j\pi/2}z^{-1})(1 - e^{-j\pi/2}z^{-1})}{(1 - 0.556z^{-1})(1 - 0.846e^{j0.3\pi}z^{-1})(1 - 0.846e^{-j0.3\pi}z^{-1})} \\
&= \frac{(1 + z^{-1})(1 + z^{-2})}{(1 - 0.556z^{-1})(1 - z^{-1} + 0.716z^{-2})} \\
&= \frac{1 + z^{-1}}{1 - 0.556z^{-1}} \cdot \frac{1 + z^{-2}}{1 - z^{-1} + 0.716z^{-2}} \\
&= \frac{1 + z^{-1}}{1 - z^{-1} + 0.716z^{-2}} \cdot \frac{1 + z^{-2}}{1 - 0.556z^{-1}} \\
&= (1 + z^{-1}) \cdot (1 + z^{-2}) \cdot \frac{1}{(1 - 0.556z^{-1})} \cdot \frac{1}{(1 - z^{-1} + 0.716z^{-2})}
\end{aligned}$$

The filter can be designed as cascaded filters: real filters will require real coefficients. So complex conjugate pole pairs are put into one filter, leading to a second order filter.