- Response to a suddenly-applied sinusoid
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1 Response to a Suddenly-Applied Sinusoid

In general, the response of an LTI filter to a suddenly-applied signal consists of two signal parts:

(a) the **transient** part

(i) the signal pattern determined by the poles of the filter

(ii) the scale factor by the poles and the input

(b) the **input-caused** part

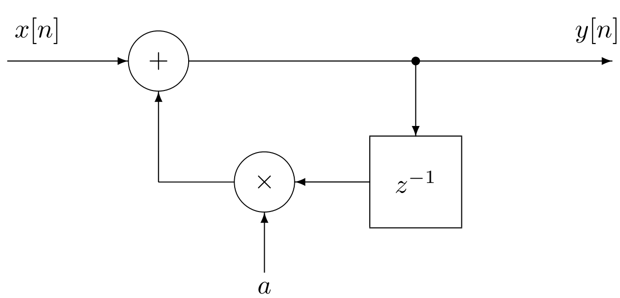
(i) the signal pattern determined by the input

(ii) the scale factor by the poles and the input

(1) (**Stable Filters**) The transient part dies out eventually.

(2) (**Unstable Filters**) The transient part does not die out.

Example 1.1 A first-order IIR filter is given in the following block diagram.



Find the response of the filter to $x[n] = K\cos(\hat{\omega}n)u[n]$.

(a) **System function:** The system function $H(z)$ is found

$$H(z) = \frac{1}{1 - az^{-1}}.$$

Note that the filter may or may not be stable depending on the value of a. It will be stable if $|a| < 1$ and unstable otherwise.

(b) **Response:** First note that

$$\cos(\hat{\omega}n) = \frac{1}{2}\big(e^{j\hat{\omega}n} + e^{-j\hat{\omega}n}\big) \qquad \Longleftrightarrow \qquad \begin{aligned} &\frac{1}{2}\Big(\frac{1}{1-e^{j\hat{\omega}}z^{-1}} + \frac{1}{1-e^{-j\hat{\omega}}z^{-1}}\Big) \\ &= \frac{1-\cos(\hat{\omega})z^{-1}}{1-2\cos(\hat{\omega})z^{-1}+z^{-2}} \\ &= \frac{z^2-\cos(\hat{\omega})z}{z^2-2\cos(\hat{\omega})z+1} \\ &= \frac{1-\cos(\hat{\omega})z^{-1}}{(1-e^{j\hat{\omega}}z^{-1})(1-e^{-j\hat{\omega}}z^{-1})}. \end{aligned}$$

The the z-transform $Y(z)$ of the output is given by

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{1}{1-az^{-1}} \cdot K\frac{1-\cos(\hat{\omega})z^{-1}}{(1-e^{j\hat{\omega}}z^{-1})(1-e^{-j\hat{\omega}}z^{-1})} \\ &= \frac{A_1}{1-az^{-1}} + \frac{A_2}{1-e^{j\hat{\omega}}z^{-1}} + \frac{A_3}{1-e^{-j\hat{\omega}}z^{-1}}, \end{aligned}$$

where

$$\begin{aligned} A_1 &= K\frac{1-\cos(\hat{\omega})z^{-1}}{(1-e^{j\hat{\omega}}z^{-1})(1-e^{-j\hat{\omega}}z^{-1})}\ \Big|_{z=a} \\ &= K\frac{1-\cos(\hat{\omega})/a}{(1-e^{j\hat{\omega}}/a)(1-e^{-j\hat{\omega}}/a)} \\ &= K\frac{1-\cos(\hat{\omega})/a}{1-2\cos(\hat{\omega})/a+1/a^2}, \\ A_2 &= \frac{K}{(1-az^{-1})(1-e^{-j\hat{\omega}}z^{-1})}\ \Big|_{z=e^{j\hat{\omega}}} \\ &= \frac{K}{(1-ae^{-j\hat{\omega}})(1-e^{-2j\hat{\omega}})}, \\ A_3 &= \frac{K}{(1-az^{-1})(1-e^{j\hat{\omega}}z^{-1})}\ \Big|_{z=e^{-j\hat{\omega}}} \\ &= \frac{K}{(1-ae^{j\hat{\omega}})(1-e^{2j\hat{\omega}})}. \end{aligned}$$

We note that $A_2 = A_3^*$ and hence

$$|A_3| = |A_2| \quad \text{and} \quad \angle A_3 = -\angle A_2.$$

Upon inverse z-transforming $Y(z)$, we get

$$\begin{aligned} y[n] &= A_1 \cdot a^n u[n] + A_2 \cdot e^{j\hat{\omega}n}u[n] + A_3 \cdot e^{-j\hat{\omega}n}u[n] \\ &= A_1 \cdot a^n u[n] + |A_2| \cdot e^{j(\hat{\omega}n+\angle A_2)}u[n] + |A_2| \cdot e^{-j(\hat{\omega}n+\angle A_2)}u[n] \\ &= \underbrace{A_1 \cdot a^n u[n]}_{\text{transient}} + \underbrace{2|A_2|\cos(\hat{\omega}n+\angle A_2)u[n]}_{\text{steady state}}. \end{aligned}$$

We note that, when a sinusoid is suddenly applied, the response has

(i) the transient part, which dies out for a stable filter,

(ii) the steady state sinusoid of the same frequency.

(c) **Conclusion:** In general, for stable filters, a suddenly-applied sinusoid causes a sinusoidal output eventually.

2 Stability of an LTI Filter and Poles

We have used the following that

An IIR filter is stable $\iff$ all poles lie inside the unit circle.

BIBO Stability

(a) (Definition) A system is BIBO-stable if the output is bounded for a bounded input, i.e., there exists a number M_Y such that

$$|y[n]| \le M_Y$$

for $|x[n]| \le M_x$.

(b) An LTI system is BIBO-stable iff

$$\sum_n |h[n]| < \infty.$$

(c) An LTI system is BIBO-stable iff all poles of $H(z)$ lie inside the unit circle.

Sketch

(i) If all poles of $H(z)$ are inside the unit circle, then $|p_k| < 1$ and from

$$\begin{aligned} H(z) &= G\frac{(z-z_1)(z-z_2)\cdots(z-z_K)}{(z-p_1)(z-p_2)\cdots(z-p_L)} \\ &= \frac{A_1}{1-p_1z^{-1}} + \frac{A_2}{1-p_2z^{-1}} + \cdots + \frac{A_L}{1-p_Lz^{-1}}, \end{aligned}$$

we get

$$h[n] = A_1p_1^n u[n] + \cdots + A_Lp_L^n u[n].$$

Then

$$\begin{aligned} \sum_n^{\infty} |h[n]| &= \sum_{n=0}^{\infty} \big|A_1p_1^n u[n] + \cdots + A_Lp_L^n u[n]\big| \\ &\le \sum_{n=0}^{\infty} \Big(\big|A_1\big| \cdot \big|p_1\big|^n + \cdots + \big|A_L\big| \cdot \big|p_L\big|^n\Big) < \infty. \end{aligned}$$

(ii) Suppose that any of the poles of $H(z)$ is outside of the unit circle. Let p_k denote the pole that has the largest magnitude, i.e., $|p_k|$ is the largest among all the poles outside the unit circle. Then, eventually as $n \to \infty$, $A_kp_k^n u[n]$ will be a dominant term of $h[n]$. That is, for large n,

$$h[n] \approx A_kp_k^n u[n].$$

Therefore,

$$|h[n]| \to \infty \quad \text{as } n \to \infty.$$

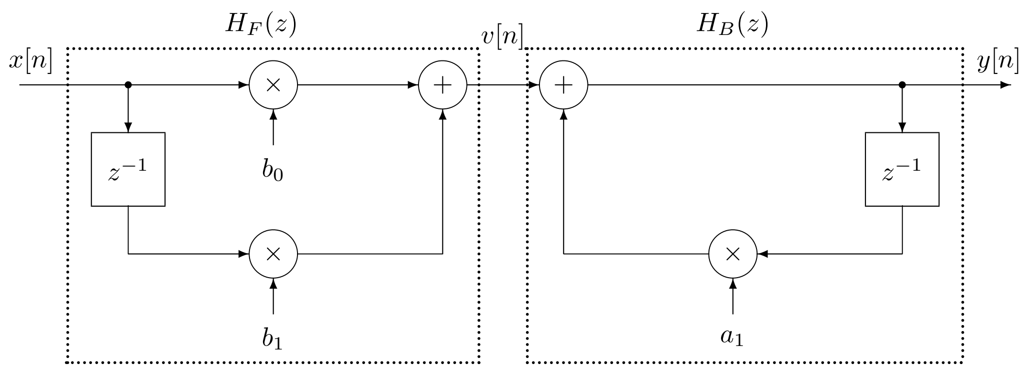
(iii) Suppose that any of the poles of $H(z)$ is on the unit circle. Let p_k denote such a pole. (Assume that it is the farthest pole from the origin.) (Note $|p_k| = 1$.) Then a bounded signal $x[n] = p_k^* u[n]$ will produce an unbounded signal:

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{k=0}^{n} A_k |p_k|^2 + \text{ other terms} \\ &= A_k(n+1) + \text{ other terms.} \end{aligned}$$

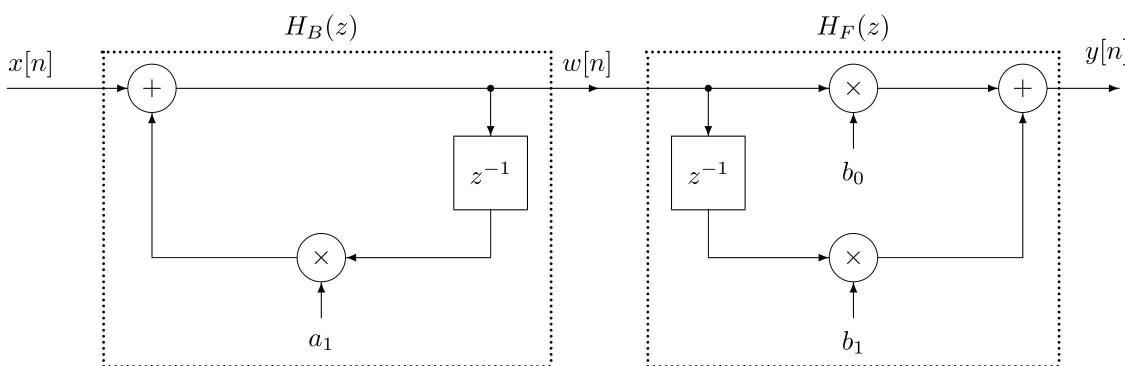
Therefore, $|y[n]| \to \infty$ as $n \to \infty$.

3 Implementation Structures

(a) **Direct Form I:** feedforward loop followed by the feedback loop



(b) Intermediate Form



(c) **Direct Form II**