Appendix A: Complex-Valued Signals

Complex-valued signals will be introduced in Chapter 2 as a way to simplify certain calculations involving sinusoidal signals. This appendix briefly summarizes the properties, statistics of complex signals and elementary operations on them.

Definition: A complex-valued signal is simply a signal whose values at each time are complex. As such, it has a real part and an imaginary part, a magnitude and a phase. For example, if

$$z(t) = x(t) + j\,y(t) = r(t)\,e^{j\phi(t)} ,$$

then $x(t)$ is the real part, $y(t)$ is the imaginary part, $r(t)$ is the amplitude and $\phi(t)$ is the angle or phase.

Signal Characteristics and Statistics:

The following table shows the definitions of the signal characteristics mentioned previously for real-valued signals, with the exception of signal value distribution, which is not easily summarized in table form.

	Continuous-time signal z(t)	**Discrete-time signal z[n]**
	$z(t) = x(t) + j\ y(t)$	**$z[n] = x[n] + j\ y]n]$**
support interval	$[t_1,t_2]$	$\{n_1,n_1+1,...,n_2\}$
duration	t_2-t_1	n_2-n_1+1
mean value:	$M(z) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} z(t)\ dt$ $= M(x) + j\,M(y)$	$M(z) = \frac{1}{n_2-n_1+1}\sum_{n=n_1}^{n_2} z[n]$ $= M(x) + j\,M(y)$
magnitude:	$\|z(t)\| = \sqrt{x^2(t)+y^2(t)}$	$\|z[n]\| = \sqrt{x^2[n]+y^2[n]}$
squared value, aka instantaneous power:	$\|z(t)\|^2 = x^2(t)+y^2(t)$	$\|z[n]\|^2 = x^2[n]+y^2[n])$
mean-squared value, aka average power:	$MS(z) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} \|z(t)\|^2\ dt$ $= MS(x) + MS(y)$	$MS(z) = \frac{1}{n_2-n_1+1}\sum_{n=n_1}^{n_2} \|z[n]\|^2$ $= MS(x) + MS(y)$
RMS value:	$RMS(z) = \sqrt{MS(z)}$	$RMS(z) = \sqrt{MS(z)}$
energy:	$E(z) = \int_{t_1}^{t_2} \|z(t)\|^2\ dt$ $= E(x) + E(y)$	$E(x) = \sum_{n=n_1}^{n_2} \|z[n]\|^2$ $= E(x) + E(y)$

Periodicity of continuous-time signals:

A complex continuous-time signal $z(t)$ is said to be *periodic with period* T if $z(t+T) = z(t)$ for all values of t. This is equivalent to saying that both $x(t)$ and $y(t)$ are periodic with period T.

1. A continuous-time signal $z(t)$ with period T is also periodic with period nT for any positive integer n.
2. The *fundamental period* T_o is the smallest period. The reciprocal of T_o is called the *fundamental frequency* f_o of the signal. That is, $f_o = 1/T_o$.
3. $z(t)$ is periodic with period T if and only if T is an integer multiple of T_o.

4. If signals z(t) and z'(t) are both periodic with period T, then the sum of these two signals, w(t) = z(t) + z'(t) is also periodic with period T. This same property holds when three or more signals are summed.

5. The sum of two signals with fundamental period T_o is periodic with period T_o, but its fundamental period might be less than T_o.

6. The sum of two signals with differing fundamental periods, T_1 and T_2, will be periodic when and only when the ratio of their fundamental periods equals the ratio of two integers. The fundamental period of the sum is the least common multiple of T_1 and T_2. The fundamental frequency of the sum is the greatest common divisor of the fundamemental frequencies of the two sinusoids.

Periodicity of discrete-time signals:

A complex discrete-time signal z[n] is said to be *periodic with period* N if z[n+N] = z[n] for all integers n. This is equivalent to saying that both x[n] and y[n] are periodic with period N.

1. A discrete-time signal with period N is also periodic with period mN for any positive integer m.

2. The *fundamental period*, denoted N_o, is the smallest period. The reciprocal of N_o is called the *fundamental frequency* f_o of the signal. That is, $f_o = 1/N_o$.

3. z[n] is periodic with period N if and only if N is an integer multiple of N_o.

4. If signals z[n] and z'[n] are both periodic with period N, then the sum of these two signals, w[n] = z[n] + z'[n] is also periodic with period N. This same property holds when three or more signals are summed.

5. The sum of two signals with fundamental period N_o is periodic with period N_o, but its fundamental period might be less than N_o.

6. The sum of two signals with differing fundamental periods, N_1 and N_2, is periodic with fundamental period equal to the least common multiple of N_1 and N_2 and fundamental frequency equal to the greatest common divisor of their fundamental frequencies f_1 and f_2. Note that unlike continuous-time case, the ratio of the fundamental periods of discrete-time periodic signals is always the ratio of two integers. Therefore, the sum is always periodic.

Elementary Operations on One Signal

These are illustrated for continuous-time signals, but apply equally to discrete-time signals.

Adding a constant: z'(t) = z(t) + c , where c is a real or complex number.

Amplitude scaling: z'(t) = c z(t) , where c is a real or complex number.

This has the effect of scaling both the average and the mean-squared values. Specifically, M(z') = c M(z) and $MS(z') = |c|^2 MS(z)$.

Time shifting: If z(t) is a signal and T is some number, then the signal

$$z'(t) = z(t-T) = x(t-T) + j\, y(t-T)$$

is a *time-shifted* version of x(t).

Time reflection/reversal: The time reflected or time reversed version of a signal z(t) is

$$z'(t) = z(-t).$$

Time scaling: The operation of *time-scaling* a signal x(t) produces a signal

$$z'(t) = z(ct)$$

where c is some positive real-valued constant.

Combinations of the above operations: In the future we will frequently encounter signals obtained by combining several of the operations introduced above, for example,

$$z'(t) = 3\ z(-2(t-1))\ .$$

Elementary Operations on Two or More Signals

These are illustrated for continuous-time signals, but apply equally to discrete-time signals.

Summing: $w(t) = z(t) + z'(t)$.

Linear combining: $w(t) = c_1 z_1(t) + c_2 z_2(t) + c_3 z_3(t)$, where c_1,c_2,c_3 are real or complex numbers.

Multiplying: $w(t) = z(t)\ z(t)$.

Concatenating: *Concatenation* is the process of appending one signal to the end of another.

Correlation

The correlation between continuous-time complex signals z(t) and z'(t) is

$$C(z,z') = \int_{t_1}^{t_2} z(t)\, z'^*(t)\ dt\ ,$$

where (t_1,t_2) is the time interval of interest. Similarly, the correlation between discrete-time complex signals z[n] and z'[n] is defined to be

$$C(z,z') = \sum_{n_1}^{n_2} z[n]\, z'^*[n]\ .$$

Why the complex conjugate? The reason is that this enables the relation E(z) = C(z,z) continue to be valid. Specifically,

$$C(z,z) = \int_{t_1}^{t_2} z(t)\, z^*(t)\ dt = E(z)\ .$$

Unfortunately, correlation for complex-valued signals is not symmetric, i.e. $C(z,z') \neq C(z',z)$. However,

$$C(z',z) = C^*(z,z')\ .$$

This is because

$$C(z',z) = \int_{t_1}^{t_2} z'(t)\, z^*(t)\ dt = \left(\int_{t_1}^{t_2} z(t)\, z'^*(t)\ dt\right)^* = C^*(z,z')\ .$$

The normalized correlation between signals z and z' is

$$C_N(z,z') = \frac{C(z,z')}{\sqrt{E(z)}\sqrt{E(z')}}\ .$$

The Schwarz Inequality continues to hold for complex signals. That is,

$$|C_N(z,z')| \leq 1,$$

with equality if and only if one signal is an amplitude scaling of the complex conjugate of the other; i.e. $y(t) = c\,x(t)$ for some real or complex constant c.

Appendix B: Trigonometric Identities and Facts About Complex Exponentials

Trigonometric Identities

We will not use these much, but nevertheless it is nice to have a table. The first five comprise Table 2.2 on p. 14 of *DSP First*.

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\cos 2\theta = \cos^2\theta - \sin^2\theta$
3. $\sin 2\theta = 2\sin\theta\cos\theta$
4. $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
5. $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
6. $\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$
7. $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$
8. $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$
9. $\cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta)$
10. $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$
11. $\sin\alpha - \sin\beta = 2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$
12. $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$
13. $\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$
14. $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$
15. $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
16. $\sin\theta = \cos(\theta-\frac{\pi}{2})$
17. $\cos\theta = \sin(\theta+\frac{\pi}{2})$

Useful Facts About Complex Exponentials

1. $e^{j\theta} = \cos\theta + j\sin\theta$ (Euler's formula)

2. $\cos\theta = \frac{1}{2}\left(e^{j\theta}+e^{-j\theta}\right)$ (Inverse Euler formula)

3. $\sin\theta = \frac{1}{2j}\left(e^{j\theta}-e^{-j\theta}\right)$ (Another Inverse Euler formula)

4. $1 = e^{j2\pi} = e^{j2\pi n}$ for any integer n

5. $-1 = e^{j\pi} = e^{-j\pi}$

6. $(-1)^n = e^{j\pi n}$

7. $j = e^{j\pi/2}$

8. $-j = \frac{1}{j} = e^{-j\pi/2}$

Appendix C: Problem Solving TIps

Simple Proof Techniques

Starting with the definition

write down what you are trying to do

write the formula you are going to use before you use it

write down partial

write neatly. you can't check what you can't read

work from both ends

write more, it saves time, you can check your reasoning/answer

give examples