Lectures on Spectra of Discrete-Time Signals

Principal questions to be addressed:

- What, in a general sense, is the "spectrum" of a discrete-time signal?
- How does one assess the spectrum of a discrete-time signal?

Outline of Coverage of the Spectra of Discrete-Time Signal

A. Introduction to the spectrum of discrete-time signals

B. Periodicity of discrete-time sinusoids and complex exponentials

C. The spectrum of a signal that is a sum of sinusoids

D. The spectrum of a periodic signal via the discrete Fourier transform

E. The spectra of segments of signals and of aperiodic signals

F. The relationship between the spectrum of a continuous-time signal and that of its samples

G. Bandwidth

Notes:

- As with continuous-time spectra, discrete-time spectra plays two important roles:
  - Analysis and design: The spectra is a theoretical tool that enables one to understand, analyze, and design signals and systems.
  - System component: The computation and manipulation of spectra is a component of many important systems.
- Presumably, the motivation for spectrum was well established in the discussion of continuous-time signal and doesn't need much further discussion here.
- It is important to stress the similarity of the spectral concept for discrete-time signals to that for continuous-time signals.

Text Material

These lecture notes are intended to serve as text material for this section of the course. Though there is some discussion in Chapter 9 about the spectrum of discrete-time signals. However, it is not required or recommended reading, it does not give a general introduction to the concept of spectrum and introduces the DFT via a frequency-bank approach, which is very different than the Chapter 3 approach to Fourier series and to our approach to the DFT. Moreover, the DFT formulas in Chapter 9 differ by a scale factor from those that we use here and in the laboratory assignments.

Lectures on Spectra of Discrete-Time Signals

These lectures introduce the concept for spectra of discrete-time signals with an-as-similar-as-possible-to-continuous-time-spectra approach.

A. Rough definition of spectrum and motivation for studying spectrum

A.1. Introduction to the concept of "spectrum"?

This introduction parallels the introduction to spectrum for continuous-time signals

Definition:

Roughly speaking, the "spectrum" of a discrete-time signal indicates how the signal may be thought of as being composed of discrete-time complex exponentials. (Note that for brevity we have jumped right to complex exponentials, rather than first indicating that we are interested in how signals are composed of sinusoids and subsequently splitting each sinusoid into two complex exponentials.)

The spectrum describes the frequencies, amplitudes and phases of the discrete-time complex exponentials that combine to create the signal.

The individual complex exponentials that sum to give the signal are called "complex exponential components".

Alternatively, the spectrum describes distribution of amplitude and phase vs. frequency of the complex exponential components.

Pairs of exponentials sum to form sinusoids.

Sinusoidal and complex exponential components are also called "spectral components".

Plotting the spectra

We like to plot and visualize spectra. We plot lines at the frequencies of the exponential components. The height of the line is the magnitude of the component. We label the line with the complex amplitude of the component, e.g. with $2e^{j.5}$.

Alternatively, sometimes we make two line plots, one showing the magnitudes of the components and the other showing the phases. These are called the "magnitude spectrum" and "phase spectrum", respectively.

A.2. Why are we interested in the spectra of discrete-time signals?

We are interested in the spectra of discrete-time signals for all the reasons that we are interested in the spectra of continuous-time signals. Presumably this doesn't require further discussion. However, the importance of spectra will be implicitly emphasized by the continued discussion and by continued examples of its application.

A.3. How does one assess the spectrum of a given signal?

As with continuous-time signals ...

- There is no single answer, i.e. there is no universal spectral concept in wide use.
- The answer/answers do not fit into one course. We begin to address this question in EECS 206. The answer continues in EECS 306 and beyond.
- We use different methods to assess the spectrum of different types of signals.

  Specifically, in this section of the course, we will discuss

  - The spectrum of a sum of sinusoids (with support $(-\infty,\infty)$)
  - The spectrum of a periodic signal (with support $(-\infty,\infty)$) via the discrete-time Fourier series, which will be called the "Discrete Fourier Transform" (DFT).
  - The spectrum of a segment of a signal via the DFT, which leads to:
    - The spectrum of an aperiodic[1] signal with finite support
    - The spectrum of an aperiodic signal with infinite support via the DFT applied to successive segments
  - The relationship of the spectrum of a continuous-time signal to the spectrum of its samples
- We won't discuss
  - The spectrum of a signal with infinite support and finite energy via the discrete-time Fourier transform (which is not the same as the DFT). This may be discussed in EECS 306.

[1] 'Aperiodic' means not periodic.

B. Periodicity of discrete-time sinusoids and complex exponentials

Before discussing discrete-time spectra, we need to discuss a couple of issues related to the periodicity of discrete-time sinusoids and complex exponentials. There are a few wrinkles in discrete time that do not happen in continuous time.

B.1 Discrete-time sinusoids

The general discrete-time sinusoid is $A\cos(\hat{\omega}n + \phi)$

- A is the amplitude. $A \geq 0$
- ϕ is the phase. As with continuous-time signals, phase ϕ and phase $\phi+2\pi$ are "equivalent" in the sense that $A\cos(\hat{\omega}n + \phi) = A\cos(\hat{\omega}n + \phi + 2\pi)$ for all n.
- $\hat{\omega}$ is the frequency. Its units are radians per sample. One could also write the sinusoid as $A\cos(fn + \phi)$, where f is frequency in cycles per sample.

  Each increment in time n increases $\hat{\omega}n$ by $\hat{\omega}$ radians
- It is generally assumed that $\hat{\omega} \geq 0$.
- We will soon see that in discrete-time, some sinusoids are not periodic, and there are "equivalent" frequencies.

Periodicity of discrete-time sinusoids

Fact B1: $A\cos(\hat{\omega}n + \phi)$ is periodic when and only when $\hat{\omega}$ is 2π times a rational number, i.e. 2π times the ratio of two integers.

If the rational number is reduced so that the numerator and denominator have no common factors except 1, then the fundamental period is the denominator of the rational number.

In contrast, recall that for continuous-time signals, every sinusoid is periodic, and the fundamental period is simply the reciprocal of the frequency in Hz.

Derivation: Recall the definition of periodicity:

x[n] is "periodic with period N" if

$$x[n+N] = x[n] \quad \text{for all } n \quad (n \text{ is an integer})$$

The "fundamental period" N_o is the smallest such period.

Let us apply the definition to see when a discrete-time sinusoid is periodic. Specifically we want to know when there is an N such that

$$A\cos(\hat{\omega}(n+N) + \phi) = A\cos(\hat{\omega}n + \phi) \quad \text{for all } n$$

Since $A\cos(\hat{\omega}(n+N) + \phi) = A\cos(\hat{\omega}n+\hat{\omega}N + \phi)$, we see that

$$A\cos(\hat{\omega}(n+N) + \phi) = A\cos(\hat{\omega}n + \phi),$$

when and only when

$$\hat{\omega}N = \text{integer} \times 2\pi,$$

or equivalently, when and only when

$$\hat{\omega} = \frac{\text{integer}}{N} \times 2\pi$$

In other words, $\hat{\omega}$ must be a rational number times 2π.

Let us now find the fundamental period of $A\cos(\hat{\omega}n + \phi)$. If the sinusoid is periodic, then $\hat{\omega} = 2\pi \frac{K}{L}$ for some integers K and L. In this case, the sinusoid is periodic with period $N = L$ or $2L$ or $3L$ or ..., because for any such value of N, $\hat{\omega}N = 2\pi \frac{K}{L} N$ is a multiple of 2π.

What is the smallest period? If we elimnate any common factors of K and L, we can write $\hat{\omega} = 2\pi \frac{K'}{L'}$, where K' and L' have no common factors except 1. By the same argument as before, $A\cos(\hat{\omega}n + \phi)$ is periodic with period L'. This is the smallest possible period, so it is the fundamental period.

Examples:

(a) $A\cos(2\pi \frac{1}{2} n)$ is periodic with frequency $\frac{1}{2}$ and fundamental period 2

(b) $A\cos(2\pi \frac{3}{5} n)$ is periodic with frequency $\frac{3}{5}$ and fundamental period 5

Notice that (b) has higher frequency, but a longer fundamental period a!
This could not happen with continuous-time signals.

(c) $A\cos(2\pi \frac{4}{5} n)$ is periodic with frequency $\frac{4}{5}$ and fundamental period 5.

Notice that (b) and (c) have different frequencies, but the same fundamental period.
This could not happen with continuous-time signals.

(d) $A\cos(2n)$ is not periodic because $\hat{\omega} = 2$ is not a rational multiple of 2π

(e) $A\cos(1.6\pi n)$ is periodic with fundamental period 5, because

$\hat{\omega} = 1.6\,\pi = 2\pi\,(0.8) = 2\pi \frac{4}{5}$ = rational multiple of 2π

fundamental period = 5

Equivalent Frequencies

Recall that phase ϕ and phase $\phi+2\pi$ are "equivalent" in the sense that

$$A\cos(\hat{\omega}n + \phi) = A\cos(\hat{\omega}n + \phi+2\pi) \quad \text{for all } n$$

As we now demonstrate, in the case of discrete-time sinusoids, there are also equivalent frequencies.

Fact B2: Frequency $\hat{\omega}$ and frequency $\hat{\omega}+2\pi$ are "equivalent" in the sense that

$$A\cos(\hat{\omega}n + \phi) = A\cos((\hat{\omega}+2\pi)n + \phi) \quad \text{for all } n$$

Derivation: For any n,

$$A\cos((\hat{\omega}+2\pi)n + \phi) = A\cos(\hat{\omega}n + 2\pi n + \phi) = A\cos(\hat{\omega}n + \phi)$$

This is another phenomena that is different for discrete time than for continuous time.

B.2 Complex exponentials

The general discrete-time complex exponential is $A e^{j\phi} e^{j\hat{\omega} n}$

- A is the amplitude. $A \geq 0$,
- ϕ is the phase. Phase ϕ and phase $\phi+2\pi$ are equivalent in the sense that

$$A e^{j\phi} e^{j\hat{\omega} n} = A e^{j(\phi+2\pi)} e^{j\hat{\omega} n}$$

- $\hat{\omega}$ is the frequency. Its units are radians per sample. One could also write the exponential as $A e^{j\phi} e^{jfn}$, where f is frequency in cycles per sample.

  We allow $\hat{\omega}$ to be positive or negative. This is because we like to think of a cosine as being the sum of complex exponentials with positive and negative frequencies.

$$A \cos(\hat{\omega} n + \phi) = \frac{A}{2} e^{j\phi} e^{j\hat{\omega} n} + \frac{A}{2} e^{-j\phi} e^{-j\hat{\omega} n}$$

Periodicity of discrete-time exponentials

Below, we list the periodicity properties of discrete-time exponentials. They are the same as discussed prevously for discrete-time sinusoids.

Fact B3: $A e^{j\phi} e^{j\hat{\omega} n}$ is periodic when and only when $\hat{\omega}$ is 2π times a rational number, i.e. 2π times the ratio of two integers.

If $\hat{\omega} = 2\pi \frac{K}{L}$, where K and L have no common factor, then the fundamental period is L.

Fact B4: Frequency $\hat{\omega}$ and frequency $\hat{\omega}+2\pi$ are equivalent in the sense that

$$A e^{j\phi} e^{j(\hat{\omega}+2\pi)n} = A e^{j\phi} e^{j\hat{\omega} n}$$

Derivation: This is because

$$A e^{j\phi} e^{j(\hat{\omega}+2\pi)n} = A e^{j\phi} e^{j\hat{\omega} n} e^{j2\pi n} = A e^{j\phi} e^{j\hat{\omega} n}$$

Discussion:

What do we make of the surprising fact that frequency $\hat{\omega}$ and frequency $\hat{\omega}+2\pi$ are equivalent? We conclude that when we consider discrete-time sinusoids or complex exponentials, we can restrict frequencies to an interval of width 2π.

- Sometimes people restrict attention to $[-\pi,\pi]$.
- Sometimes people restrict attention to $[0,2\pi]$.
- We'll do a bit of both.

C. The spectrum of a finite sum of discrete-time sinusoids

Our discussion of how to assess a spectrum parallels the discussion for continuous-time sinusoids. We begin by considering signals that are finite sums of sinsusoids. However, because of the possibility of equivalent frequencies, there are a couple of key differences in how discrete-time and continuous-time spectra are assessed.

C.1. Example

Let us start with an example of a signal that is a finite sum of sinusoids:

$$x[n] \;=\; 2\cos(\tfrac{1}{2}\pi n+.1) - 4\cos(\tfrac{3}{4}\pi n+.1) + 2\cos(\tfrac{3}{2}\pi n+.1) + 2\cos(\tfrac{5}{2}\pi n+.1)$$

Following the approach for continuous-time sepctra, we first decompose $x[n]$ into a sum of complex exponentials:

$$x[n] \;=\; e^{j(.1)}\, e^{j\frac{1}{2}\pi n} \;+\; 2\,e^{j(.1+\pi)}\, e^{j\frac{3}{4}\pi n} \;+\; e^{j(.1)}\, e^{j\frac{3}{2}\pi n} \;+\; e^{j(.1)}\, e^{j\frac{5}{2}\pi n}$$

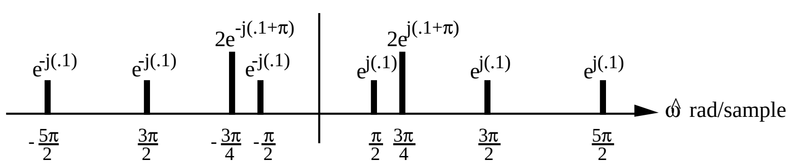
$$+\, e^{-j(.1)}\, e^{-j\frac{1}{2}\pi n} + 2\,e^{-j(.1+\pi)}\, e^{-j\frac{3}{4}\pi n} + e^{-j(.1)}\, e^{-j\frac{3}{2}\pi n} + e^{-j(.1)}\, e^{-j\frac{5}{2}\pi n}$$

It would now seem natural to identify the spectrum as the set of complex amplitude and frequency pairs:

$$\Big\{(e^{-j(.1)}, -\tfrac{5}{2}\pi),\ (e^{-j(.1)}, -\tfrac{3}{2}\pi),\ (2\,e^{-j(.1+\pi)}, -\tfrac{3}{4}\pi),\ (e^{-j(.1)}, -\tfrac{1}{2}\pi),$$

$$(e^{j(.1)}, \tfrac{1}{2}\pi),\ (2\,e^{j(.1+\pi)}, \tfrac{3}{4}\pi),\ (e^{j(.1)}, \tfrac{3}{2}\pi),\ (e^{j(.1)}, \tfrac{5}{2}\pi)\Big\}$$

and to draw the spectrum as shown below:



However, some of these exponentials have equivalent frequencies, causing the above plot to be misleading. Specifically,

- frequencies $-\frac{3}{2}\pi,\ \frac{1}{2}\pi,\ \frac{5}{2}\pi$ are equivalent because they differ by 2π or a multiple of 2π.
- frequencies $-\frac{5}{2}\pi,\ -\frac{1}{2}\pi,\ \frac{3}{2}\pi$ are equivalent for the same reason

Therefore, we combine all exponentials with equivalent frequencies (using phasor addition) into a single exponential component. In doing so, we get to choose which of the equivalent frequencies the resulting exponential component will have. There are two possible conventions:

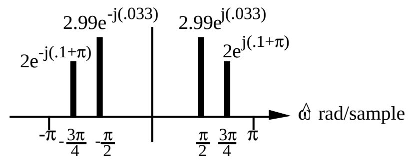
- Two-sided spectra: In this case, the exponential components have frequencies in the interval $(-\pi,\pi]$.
- One-sided spectra: In this case, the exponential components have frequencies in the interval $[0,2\pi)$.

Choosing between these two conventions is mainly a matter of taste.

- If we choose the two-sided convention, then the spectrum of $x[n]$ is

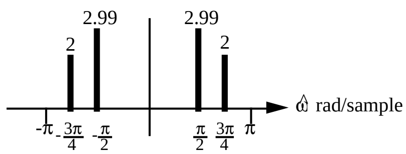
$$\left\{(2e^{-j(.1+\pi)}, -\tfrac{3}{4}\pi),\ (2.99e^{-j(.033)}, -\tfrac{1}{2}\pi),\ (2.99e^{j(.033)}, \tfrac{1}{2}\pi),\ (2e^{j(.1+\pi)}, \tfrac{3}{4}\pi)\right\}$$

where, for example, $2.99e^{j(.033)} = e^{-j(.1)} + e^{j(.1)} + e^{j(.1)}$. This spectrum is plotted below

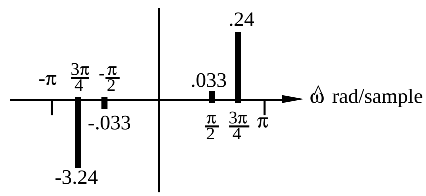


Alternatively, the magnitude and phase spectra are shown below.

magnitude spectra



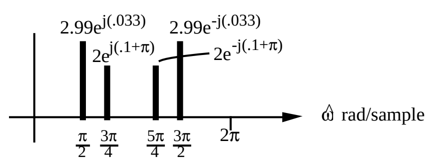
phase spectra



- If we choose the one-sided convention, then the spectrum of $x[n]$ is

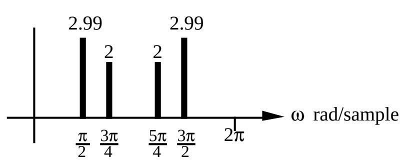
$$\left\{(2.99e^{j(.033)}, \tfrac{1}{2}\pi),\ (2e^{j(.1+\pi)}, \tfrac{3}{4}\pi),\ (2e^{-j(.1+\pi)}, \tfrac{5}{4}\pi),\ (2.99e^{-j(.033)}, \tfrac{3}{2}\pi),\ \right\}$$

as plotted below:

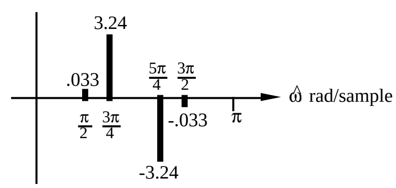


Alternatively, the magnitude and phase spectra are shown below:

magnitude spectra



phase spectra



C.2. Spectrum of a general sum of discrete-time sinusoids

More generally, consider a signal of the form

$$x[n] = A_0 + \sum_{k=1}^{N} A_k \cos(\hat{\omega}_k n + \phi_k)$$

$$= A_0 + A_1\cos(\hat{\omega}_1 n+\phi_1) + A_2\cos(\hat{\omega}_2 n+\phi_2) + \ldots + A_N\cos(\hat{\omega}_N t+\phi_N)$$

where N, A_0, A_1, $\hat{\omega}_1$, ϕ_1, ..., A_N, $\hat{\omega}_N$, ϕ_N are parameters that specify $x[n]$.

We now rewrite this in several ways. First, using Euler's formula, we rewrite $x[n]$ as

$$x[n] = X_0 + \sum_{k=1}^{N} \mathrm{Re}\left\{X_k\, e^{j\hat{\omega}_k n}\right\}$$

where

$$X_k = A_k e^{j\phi_k}$$

is the phasor corresponding to $A_k \cos(\hat{\omega}_k n+\phi)$. ($X_k$ is a complex number.)

Second, using the inverse Euler formula, we rewrite this

$$x[n] = X_0 + \sum_{k=1}^{N} \left(\frac{X_k}{2} e^{j\hat{\omega}_k n} + \frac{X^*_k}{2} e^{-j\hat{\omega}_k n} \right),$$

which can also be written as

$$x[n] = \sum_{k=-N}^{N} \beta_k e^{j\hat{\omega}_k n}$$

where $\beta_0 = X_0 = A_0$

$$\beta_k = \begin{cases} \frac{X_k}{2}, & k \geq 1 \\ \frac{X^*_k}{2}, & k \leq -1 \end{cases}$$

Finally, we combine terms with equivalent frequencies, to obtain

$$x[n] = \sum_{k=-N'}^{N'} \alpha_k e^{j\hat{\omega}_k n}$$

where $N' \leq N$ and where each $\hat{\omega}_k$ is between $-\pi$ and π. Note that $\alpha_{-k} = \alpha^*_k$, $|\alpha_{-k}| = |\alpha_k|$, and $\text{angle}(\alpha_{-k}) = -\text{angle}(\alpha_k)$.

- The two-sided spectrum is, then,

$$\left\{ (\alpha_{-N'},\hat{\omega}_{-N'}), ..., (\alpha_{-1},\hat{\omega}_{-1}), (\alpha_0,0), (\alpha_1,\hat{\omega}_1), ..., (\alpha_{N'},\hat{\omega}_{N'}) \right\}$$

$$= \left\{ (\alpha^*_{N'},-\hat{\omega}_{N'}), ..., (\alpha^*_1,-\hat{\omega}_1), (\alpha_0,0), (\alpha_1,\hat{\omega}_1), ..., (\alpha_{N'},\hat{\omega}_{N'}) \right\}$$

- The one-sided spectrum is

$$\left\{ (\alpha_0,0), (\alpha_1,\hat{\omega}_1), ..., (\alpha_{N'},\hat{\omega}_{N'}), (\alpha_{-N'},2\pi+\hat{\omega}_{-N'}), ..., (\alpha_{-1},2\pi+\hat{\omega}_{-1}) \right\}$$

$$= \left\{ (\alpha_0,0), (\alpha_1,\hat{\omega}_1), ..., (\alpha_{N'},\hat{\omega}_{N'}), (\alpha^*_{N'},2\pi-\hat{\omega}_{N'}), ..., (\alpha^*_1,2\pi-\hat{\omega}_1) \right\}$$

As with continuous-time spectra,

- The spectrum, i.e. one of these lists, is considered to be a simpler more compact representation of the signal $x[n]$, i.e. just a few numbers.
- The "spectrum" is often called the "frequency-domain representation" of the signal. In contrast, $x[n]$ is called the "time-domain representation" of the signal.
- The term $\alpha_k e^{j\hat{\omega}_k n}$ is called the "complex exponential component" or "spectral component" of $x[n]$ at frequency $\hat{\omega}_k$.
- To obtain a useful visualization, we often plot the spectrum. That is, for each k, we draw a "spectral line" at frequency $\hat{\omega}_k$ with height equal to $|\alpha_k|$, and we label the line with the value of α_k, which is in general is complex.

- Alternatively, we sometimes separate the spectrum into magnitude and phase parts. For example, the two-sided versions of these are shown below.
  - Magnitude spectrum

$$\left\{(|\alpha_{-N'}|,\hat{\omega}_{-N'}),\ \ldots,\ (|\alpha_{-1}|,\hat{\omega}_{-1}),\ (|\alpha_0|,0),\ (|\alpha_1|,\hat{\omega}_1),\ \ldots,\ (|\alpha_{N'}|,\hat{\omega}_{N'})\right\}$$

$$= \left\{(|\alpha_{N'}|,\hat{\omega}_{-N'}),\ \ldots,\ (|\alpha_1|,\hat{\omega}_{-1}),\ (|\alpha_0|,0),\ (|\alpha_1|,\hat{\omega}_1),\ \ldots,\ (|\alpha_{N'}|,\hat{\omega}_{N'})\right\}$$

  - Phase spectrum

$$\left\{(\text{angle}(\alpha_{-N'}),\hat{\omega}_{-N'}),\ \ldots,\ (\text{angle}(\alpha_{-1}),\hat{\omega}_{-1}),\ (\text{angle}(\alpha_0),0),\ (\text{angle}(\alpha_1),\hat{\omega}_1),\ \ldots,\ (\text{angle}(\alpha_{N'}),\hat{\omega}_{N'})\right\}$$

$$= \left\{(-\text{angle}(\alpha_{N'}),\hat{\omega}_{-N'}),\ \ldots,\ (-\text{angle}(\alpha_1),\hat{\omega}_{-1}),\ (\text{angle}(\alpha_0),0),\ (\text{angle}(\alpha_1),\hat{\omega}_1),\ \ldots,\ (\text{angle}(\alpha_{N'}),\hat{\omega}_{N'})\right\}$$

  And we might draw separate plots of magnitude and phase. That is, for each k, the magnitude plot has a line of height $|\alpha_k|$ at frequency $\hat{\omega}_k$, and the phase plot has a line of height angle(α_k) at frequency $\hat{\omega}_k$.

- Often, but certainly not always, we are more interested in the magnitude spectrum than the phase spectrum.

Example: Given that the spectrum of the signal x[n] is shown below, find x[n].

Plot of spectrum here.

Pick off and sum the exponential components. Simplify so as to write x[n] as a sum of sinusoids in standard form.

Plot x[n].

D. The spectrum of a periodic discrete-time signal

Recall that from the Fourier series theorem we learned that the spectrum of a periodic continuous-time signal with period T is concentrated at frequencies that are multiples of 1/T, and that the Fourier series analysis formula determines the specific component at each of these frequencies. In this section we learn of an analogous theorem that indicates that the spectrum of a periodic discrete-time signal with period N is concentrated at frequencies that are multiples of 1/N and that provides an analysis formula for determining the specific component at each of these frequencies.

D.1 The DFT Theorem

The spectrum of a discrete-time periodic signal derives from the following theorem.

The Discrete-Fourier Transform (DFT) Theorem
(aka The Discrete-Time Fourier Series Theorem)

A periodic signal $x[n]$ with period N can be written as a sum of N complex exponentials with frequencies $0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, ..., (N-1)\frac{2\pi}{N}$. Specifically, there are N DFT coefficients $X[0], X[1], ..., X[N-1]$ such that

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn} \qquad \text{(the synthesis formula)}$$

The DFT coefficients are determined from the signal $x[n]$ via

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad \text{(the analysis formula)}$$

Notes:

1. We will derive this theorem later. Unlike the Fourier Series Theorem, its derivation is well within the scope of this class.

2. The term $X[k]\, e^{j\frac{2\pi}{N}kn}$ appearing in the synthesis formula is the complex exponential component (equivalently, the spectral component) of $x[n]$ at frequency $\frac{2\pi}{T}k$.

3. The theorem says that ANY periodic signal can be represented as the sum of at most N complex exponential components with frequencies coming from the set

$$\left\{0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, ..., (N-1)\frac{2\pi}{N}\right\}$$

   This means that the spectrum of a periodic signal with period N is concentrated at these frequencies or a subset thereof.

4. The synthesis formula is very much like the synthesis formula for the Fourier series of a continuous-time periodic signal, except that only a finite number of frequencies/exponential components are used. This stems from the fact that for discrete-time complex exponentials, every frequency outside the range 0 to 2π is equivalent to some frequency within the range 0 to 2π.

5. For periodic signals, it is natural then to use the following as the definition of the spectrum:

$$\left\{(X[0],0),\ (X[1],\tfrac{2\pi}{N}),\ (X[2],2\tfrac{2\pi}{N}),\ ...,\ (X[N-1],(N-1)\tfrac{2\pi}{N})\right\}$$

Therefore, finding the spectrum of a periodic discrete-time signal involves finding its period and finding the $X[k]$'s.

Notice that the above defines a one-sided spectrum. We could also define a two-sided spectrum. However, as demonstrated later, the two-sided spectrum is somewhat messier.

There is no need to combine exponential components with equivalent frequencies, as we did previously when finding the spectrum of a finite sum of sinusoids. In essence, the DFT analysis formula has already done this for us.

7. To aid the understanding of the synthesis and analysis formulas, it is often useful to view them in long form:

   Synthesis formula:

$$x[n] = X[0] + X[1]\, e^{j\frac{2\pi}{N}n} + X[2]\, e^{j\frac{2\pi}{N}2n} + ... + X[N-1]\, e^{j\frac{2\pi}{N}(N-1)n}$$

   Analysis formula:

$$X[k] = \frac{1}{N}\left(x[0] + x[1]\, e^{j\frac{2\pi}{N}k} + x[2]\, e^{j\frac{2\pi}{N}k2} + \ldots + x[N-1]\, e^{j\frac{2\pi}{N}k(N-1)} \right)$$

   An elementary example of computing these formulas is given in Section D.2.

8. The frequency $\frac{2\pi}{N}$ is called the "fundamental" or "first harmonic" frequency. The frequency $k\frac{2\pi}{N}$ is called the "kth-harmonic" frequency. Likewise, the component at frequency $\frac{2\pi}{N}$ is called the "fundamental" or "first harmonic" component. The frequency $k\frac{2\pi}{N}$ is called the "kth-harmonic" component.

9. The analysis formula may be viewed as operating on a periodic signal $x[n]$ (actually, just $x[0],...,x[N-1]$) and producing N DFT coefficients $X[0],...,X[N-1]$. This operation is considered to be a "transform" of the signal $x[n]$ into the set of coefficients $X[0],...,X[N-1]$. This is why "transform" appears in the name "Discrete Fourier Transform". In a similar manner, the synthesis formula may be viewed as operating on Fourier coefficients $X[0],...,X[N-1]$ and producing a signal $x[n]$. This operation is considered to be an "inverse transform".

10. It is customary to use $X[k]$ as a shorthand for $X[0],...,X[N-1]$. Thus, as with $x[n]$, there are two possible meanings for the notation "$X[k]$": it could mean the kth coefficient, or it could mean the entire set of N coefficients.

11. The term "Discrete-Fourier Transform" (DFT) commonly refers both to the process of applying the analysis formula and to the coefficients $X[k]$ that result from this process. For example, people often say "X[k] is the DFT of x[n]".

12. The process of applying the analysis formula to $x[n]$ is often called "finding/taking the DFT of $x[n]$" or, even, "DFT'ing $x[n]$". Similarly, the process of synthesizing $x[n]$ from the DFT coefficients $X[k]$ is often called "finding/taking the inverse DFT of $X[k]$", or "inverse DFT'ing $X[k]$".

13. It is traditional to use $X[k]$ to denote the DFT of $x[n]$, $X_1[k]$ to denote the DFT of $x_1[n]$, $Y[k]$ to denote the DFT of $y[n]$, and so on.

14. For the case that N is a power of 2, e.g. $N = 2^n$, there is fast algorithm for computing the DFT, called the "Fast Fourier Transform" (FFT). This algorithm has enabled the widespread use of the DFT in the analysis, design and implementation of signals and systems.

15. Since the summand in the analysis formula is periodic with period N, the limits of the summation could be replaced with any interval of length N.

16. If a signal has period N, then it also has period 2N and 3N and so on. Thus, when applying DFT analysis, we have a choice as to N. Often, but certainly not always, we choose N to equal the fundamental period. When we want to explicitly specify the value of N used, we will say "the N-point DFT". As discussed in Section D.4, changing N to 2N has no actual effect on the spectrum.

17. If instead of a one-sided spectrum we are interested in a two-sided spectrum, then it follows from the theorem that the two-sided spectrum of a periodic signal with period N is concentrated at frequencies

$$(-\frac{N}{2}+1)\frac{2\pi}{N}, \ldots, -2\frac{2\pi}{N}, -\frac{2\pi}{N}, 0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, \ldots, \frac{N}{2}\frac{2\pi}{N} \qquad \text{when } N \text{ is even}$$ [2]

$$-\frac{(N-1)}{2}\frac{2\pi}{N}, \ldots, -2\frac{2\pi}{N}, -\frac{2\pi}{N}, 0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, \ldots, \frac{(N-1)}{2}\frac{2\pi}{N} \qquad \text{when } N \text{ is odd}$$

The two-sided spectrum is then

$$\left\{\left(X[\tfrac{N}{2}+1],(-\tfrac{N}{2}+1)\tfrac{2\pi}{N}\right), \ldots, \left(X[N-1],-\tfrac{2\pi}{N}\right), \left(X[0],0\right), \left(X[1],\tfrac{2\pi}{N}\right), \ldots, \left(X[\tfrac{N}{2}],\tfrac{N}{2}\tfrac{2\pi}{N}\right)\right\}$$

when N is even

$$\left\{\left(X[-\tfrac{(N-1)}{2}],(-\tfrac{(N-1)}{2})\tfrac{2\pi}{N}\right), \ldots, \left(X[N-1],-\tfrac{2\pi}{N}\right), \left(X[0],0\right), \left(X[1],\tfrac{2\pi}{N}\right), \ldots, \left(X[\tfrac{N}{2}],\tfrac{N}{2}\tfrac{2\pi}{N}\right)\right\}$$

when N is odd

The messiness of the two-sided spectrum is probably why the DFT is defined so as to directly give a one-sided spectrum. From now on, we will mostly use the one-sided spectrum.

18. The DFT Theorem applies to complex signals, as well as to real signals.

19. Observe that according to the analysis formula, coefficient X[k] is computed by correlating x[n] with $e^{j\frac{2\pi}{N}kn}$ and normalized by 1/N, which is the energy of one period of the exponential.

Suggested reading. The discussion of "signal components" at the end of Section IIIB of "Introduction to Signals" by DLN. In the terminology of that discussion

$X[k]\, e^{j\frac{2\pi}{N}kn}$ is the component of x[n] that is like $e^{j\frac{2\pi}{N}kn}$

X[k] measures the similarity of x[n] to the exponential.

---

[2]The decision to have frequencies extending from $(-\frac{N}{2}+1)\frac{2\pi}{N}$ to $\frac{N}{2}\frac{2\pi}{N}$, rather than from $-\frac{N}{2}\frac{2\pi}{N}$ to $(\frac{N}{2}-1)\frac{2\pi}{N}$, is arbitrary, and not a univeral convention.

20. The reader is encouraged review the discussion of Fourier series for continuous-time signals and observe the similarities with the DFT for discrete-time periodic signals. The principal differences are

   t is replaced by n

   T is replaced by N

   The DFT synthesis formula has only N terms.

   The DFT analysis formula uses a sum rather than an integral

D.2 Examples

Example 1: Let us find the spectrum of the periodic signal with period 4 and

$$x[0] = 1,\ \ x[1] = 1,\ x[2] = 0,\ \ x[3] = 0$$

Since the signal has period 4 we may choose $N = 4$. Then from the DFT analysis formula we have

$$X[0] = \tfrac{1}{4}\left(x[0] + x[1]\, e^{j\frac{2\pi}{4}0\times 1} + x[2]\, e^{j\frac{2\pi}{4}0\times 2} + x[3]\, e^{j\frac{2\pi}{4}0\times 3}\right)$$

$$= \tfrac{1}{4}(1 + 1 + 0 + 0) = \tfrac{1}{2}$$

$$X[1] = \tfrac{1}{4}\left(x[0] + x[1]\, e^{j\frac{2\pi}{4}1\times 1} + x[2]\, e^{j\frac{2\pi}{4}1\times 2} + x[3]\, e^{j\frac{2\pi}{4}1\times 3}\right)$$

$$= \tfrac{1}{4}\left(1 + e^{j\frac{\pi}{2}} + 0 + 0\right) = \tfrac{1}{4}(1+j) = \tfrac{\sqrt{2}}{4}\, e^{j\pi/4}$$

$$X[2] = \tfrac{1}{4}\left(x[0] + x[1]\, e^{j\frac{2\pi}{4}2\times 1} + x[2]\, e^{j\frac{2\pi}{4}2\times 2} + x[3]\, e^{j\frac{2\pi}{4}2\times 3}\right)$$

$$= \tfrac{1}{4}\left(1 + e^{j\pi} + 0 + 0\right) = \tfrac{1}{4}(1-1) = 0$$

$$X[3] = \tfrac{1}{4}\left(x[0] + x[1]\, e^{j\frac{2\pi}{4}3\times 1} + x[2]\, e^{j\frac{2\pi}{4}3\times 2} + x[3]\, e^{j\frac{2\pi}{4}3\times 3}\right)$$

$$= \tfrac{1}{4}\left(1 + e^{j\frac{3\pi}{2}} + 0 + 0\right) = \tfrac{1}{4}(1-j) = \tfrac{\sqrt{2}}{4}\, e^{-j\pi/4}$$

In summary,

$$X[0] = \tfrac{1}{2},\ \ X[1] = \tfrac{\sqrt{2}}{4}\, e^{j\pi/4},\ \ X[2] = 0,\ \ X[3] = \tfrac{\sqrt{2}}{4}\, e^{-j\pi/4}$$

and the spectrum is

$$\left\{ \left(\tfrac{1}{2},0\right),\ \ \left(\tfrac{\sqrt{2}}{4}\, e^{j\pi/4}, \tfrac{\pi}{2}\right),\ \ \left(\tfrac{\sqrt{2}}{4}\, e^{-j\pi/4}, \tfrac{3\pi}{2}\right) \right\}$$

The next four examples give the N-point DFT coefficients of periodic exponentials and sinsuosids. These can be found by inspection.

Example 2: An exponential with frequency that is a multiple of 1/N:

$$x[n] = e^{j(2\pi\frac{m}{N}n+\phi)} \;\Rightarrow\; X[k] = \begin{cases} e^{j\phi}, & k=m \\ 0, & k\neq m \end{cases}$$

X[k] can be computed by inspection, as in Section C.

Example 3: A cosine with frequency that is a multiple of 1/N:

$$x[n] = \cos(2\pi\frac{m}{N}n) \;\Rightarrow\; X[k] = \begin{cases} \frac{1}{2}, & k=m,\ N-m \\ 0, & \text{else} \end{cases}$$

X[k] can be computed by inspection after decomposing x[n] into complex exponentials via the inverse Euler formula, as in Section C.

Example 4: A sine with frequency that is a multiple of 1/N:

$$x[n] = \sin(2\pi\frac{m}{N}n) \;\Rightarrow\; X[k] = \begin{cases} \frac{1}{2j}, & k=m \\ -\frac{1}{2j}, & k=N-m \\ 0, & \text{else} \end{cases}$$

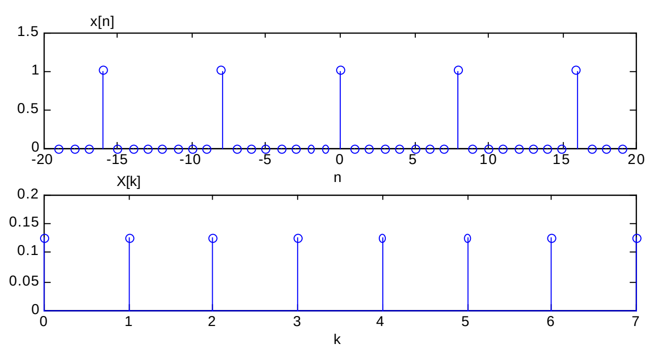
X[k] can be computed by inspection after decomposing x[n] into complex exponentials via the inverse Euler formula, as in Section C.

Example 5: A cosine with phase shift and frequency that is a multiple of 1/N

$$x[n] = \cos(2\pi\frac{m}{N}n+\phi) \;\Rightarrow\; X[k] = \begin{cases} \frac{1}{2}e^{j\phi}, & k=m \\ \frac{1}{2}e^{-j\phi}, & k=N-m \\ 0, & \text{else} \end{cases}$$

X[k] can be computed by inspection after decomposing x[n] into complex exponentials via the inverse Euler formula, as in Section C.

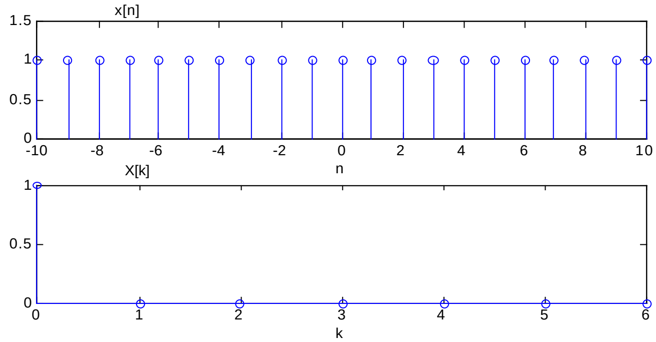
Example 6: $x[n] = \begin{cases} 1, & n=\text{multiple of } N \\ 0, & \text{else} \end{cases} \;\Rightarrow\; X[k] = \frac{1}{N},\ \ k = 0,\ldots,N-1$



(N=8)

In this case, X[k] is computed using the DFT analysis formula.

Example 7: $x[n] = 1$, all $n \Rightarrow X[k] = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$

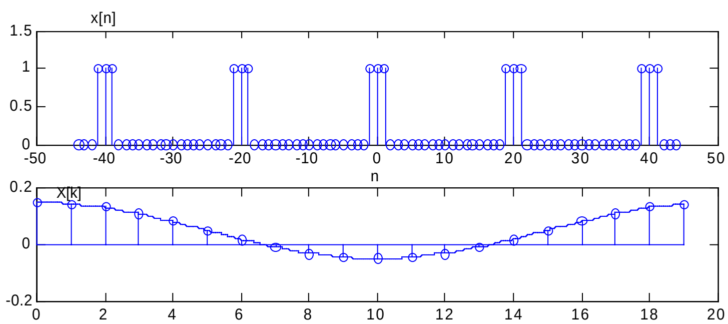


(N=7)

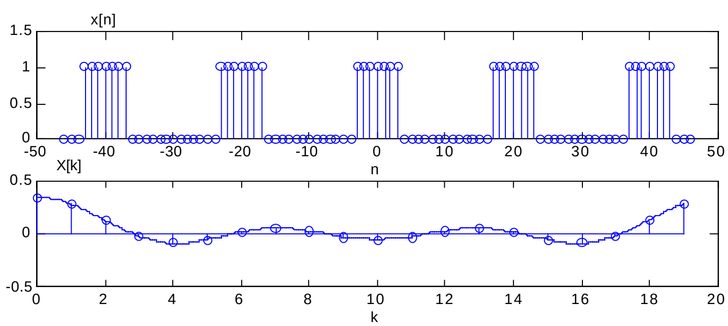
In this case, $X[k]$ is computed using the DFT analysis formula.

Example 8: $x[n]$ is periodic with fundamental period N and $x[n] = \begin{cases} 0, & -N/2 \leq n \leq -m-1 \\ 1, & -m \leq n \leq m \\ 0, & m+1 \leq n \leq N/2 \end{cases}$

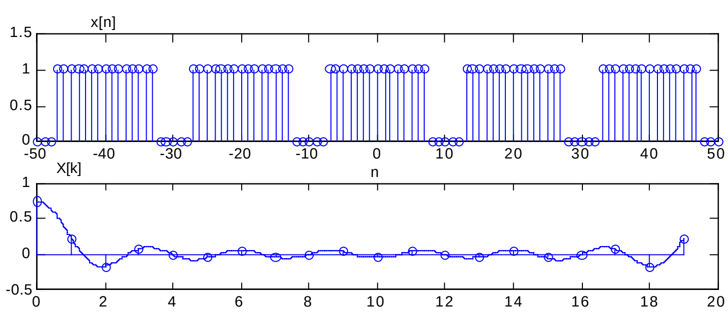
$$\Rightarrow X[k] = \frac{\sin((2m+1)\hat{\omega}/2)}{\sin(\hat{\omega}/2)} \Big|_{\hat{\omega}=2\pi k/N}$$



(N=20,m=1)



(N=20,m=3)



(N=20,m=7)

In this case, $X[k]$ is computed using the DFT analysis formula.

Note that for these examples, $X[k]$ is real valued.

The smooth curve is the "envelope" of the DFT, which depends on N but not m.

It is interesting to notice that as the number of consecutive 1' increases, the spectrum becomes more concentrated at low frequencies. Conversely, as the number of consecutive 1's decreases, the spectrum becomes more spread out in frequency. This is representative of the "rule of the thumb" that generally speaking signals that are made of short pulses have more high frequency content than signals made of long pulses.

Example 9: Find the spectrum of the following signal:

$x[n]$ = sawtooth wave

Find a closed form expression for the coefficients.

Examine what happens to the spectrum as the parameters of the signal change.

Notice that for this signal, the spectrum cannot be computed by inspection. We very much need the analysis formula.

Example 10: Find the spectrum of the following example

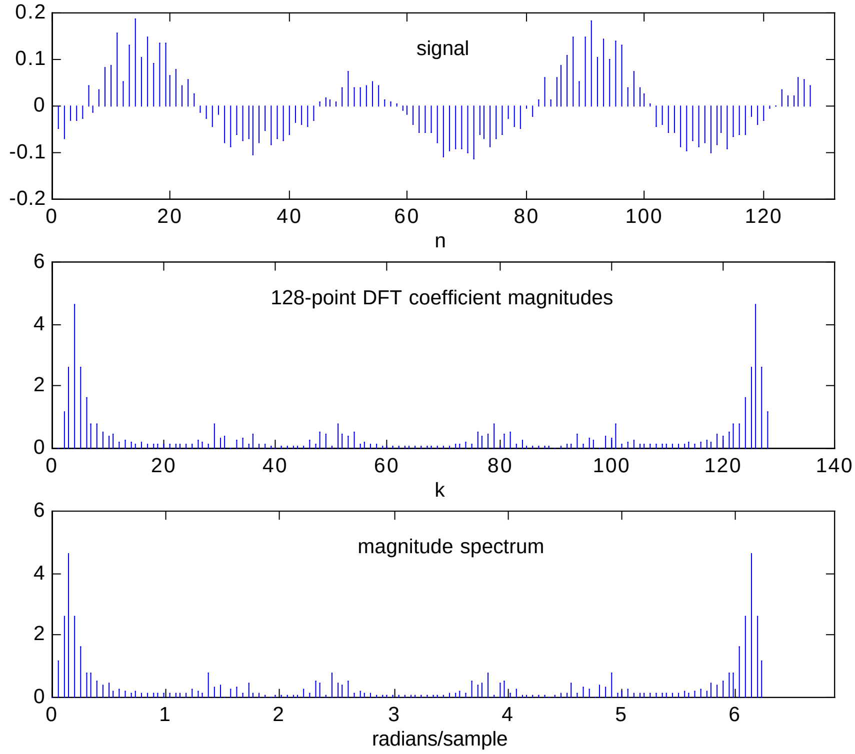
$x[n]$ = finite sum of sinusoids.

In this example one computes the spectrum (i.e. the Fourier series coefficients) by inspection just as we did in Section C.2. The one-to-oneness of the relation between Fourier coefficients and periodic signals (see Fact D1 in Section D.3 below) means that the coefficients we obtain by inspection are the Fourier series coefficients.

Example 11: Find the signal corresponding to the following spectrum.

Show a spectrum with finite number of spectral lines. This is the same sort of problem as in the section on finite sums of sinusoids section.

Example 12: The signal shown below comes from someone speaking the vowel 'e'. It is nearly periodic. The magnitudes of its 128-point DFT and its magnitude spectrum are also shown.



D.3 Derivation of the DFT

To demonstrate the validity of the theorem, we will first show that when the analysis formula for X[k] is substituted into the synthesis formula, the result is x[n]. We will then show that when the synthesis formula holds, the analysis formula is the one and only way to determine the coefficients.

These demonstrations rely on the following fact, which we will derive before demonstrating the validity of the DFT Theorem. This fact will also be useful at other times in the course.

Fact D1:

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n} = \begin{cases} N, & k=m \\ 0, & k\neq m \end{cases}$$

Derivation:

Case 1: When k = m, the exponent of each term is 0, making each term equal 1. Hence,

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n} = N$$

Case 2: When $k\neq m$, to simplify notation, let $z = e^{j\frac{2\pi}{N}(k-m)}$. Now

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n} = \sum_{n=0}^{N-1} z^n$$

which we recognize as a finite geometric series. Specifically, it is well known that

$$\sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z}$$

Using this gives

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n} = \frac{1-e^{j\frac{2\pi}{N}(k-m)N}}{1-e^{j\frac{2\pi}{N}(k-m)}} = \frac{1-e^{j2\pi(k-m)}}{1-e^{j\frac{2\pi}{N}(k-m)}} \quad \text{cancelling N's}$$

$$= \frac{1-1}{1-e^{j\frac{2\pi}{N}(k-m)}} = 0 \quad \text{since the exponent in the numerator is j times a multiple of } 2\pi$$

Derivation of the DFT Theorem:

Substituting the analysis formula for X[k] into the synthesis formulas gives

$$\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1} \left(\frac{1}{N} \sum_{n'=0}^{N-1} x[n']\, e^{-j\frac{2\pi}{N}kn'} \right) e^{j\frac{2\pi}{N}kn}$$

note that since variable n is already,used in the synthesis formula, n' is used in the analysis formula

$$= \frac{1}{N} \sum_{n'=0}^{N-1} x[n'] \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-n')} \quad \text{rearranging terms}$$

$$= \frac{1}{N}\, x[n]\, N = x[n]$$

where the next to last equality is due to Fact 1. Specifically, the rightmost sum equals 0 when the exponent is not zero, i.e. when $n' \neq n$, and equals N when the exponent is zero, i.e. when $n'=n$. Therefore, x[n'] is multipled by 0 when $n' \neq n$, and by N when $n'=n$.

Finally, we show that if the synthesis formula holds, the coefficients must be calculated via the analysis formula. Accordingly, let us assume that the synthesis formula holds, i.e.

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

Let us correlate both sides of this equation with $e^{j\frac{2\pi}{N}k'n}$. That is, we multiply both sides of the above by $\left(e^{j\frac{2\pi}{N}k'n}\right)^*$

$$x[n]\, e^{-j\frac{2\pi}{N}k'n} = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}\, e^{-j\frac{2\pi}{N}k'n}$$

and sum over values of n

$$\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}k'n} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}\, e^{-j\frac{2\pi}{N}k'n}$$

$$= \sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-k')n} \quad \text{rearranging terms}$$

$$= X[k']$$

where the last equality is due to Fact 1. Specifically, the rightmost sum equals 0 when the exponent is not zero, i.e. when $k \neq k'$, and equals N when the exponent is zero, i.e. when $k=k'$. Therefore, X[k] is multipled by 0 when $k \neq k'$, and by N when $k=k'$. This last equality demonstrates that the X[k'] must equal the synthesis formula.

D.4 Properties of the DFT

This section lists a number of useful properties of the DFT.

Property D1: There is a one-to-one relationship between periodic signals with period N and sets of DFT coefficients. Specifically, for any given signal x[n], the analysis formula gives the unique set of coefficients from which the synthesis formula yields x[n]. This means that the DFT coefficients can sometimes by found by means other than the analysis formula, e.g. inspection. That is, if by some means you find X[k] such that

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn},$$

then this X[k] is necessarily the DFT that would be computed by the analysis formula.

Similarly, for any given set of DFT coefficients X[k], the synthesis formula gives the unique signal x[n] from which the analysis formula yields X[k]. That is, if by some means you find a signal x[n] such that

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

then signal x[n] is the one and only signal having X[k] as its coefficients.

Another statement of the one-to-oneness is that if $x_1[n]$ and $x_2[n]$ are distinct signals[3], each with period N, then for at least one k, X[k] for $x_1[n]$ does not equal X[k] for $x_2[n]$.

Property D2: X[0] is the mean or DC value of x[n}.

This is because

$$X[0] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}0n} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] = m(x)$$

Property D3: One can compute the DFT coefficients by summing over any time interval of length N.

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{n=m}^{m+N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

Property D4: Conjugate symmetry (important). When the signal x[n] is real,

$$X[N-k] = X^*[k], \text{ for all } k$$

This shows that if one knows X[k] for $k \le N/2$, then one can easily find the remaining X[k]'s. Note that X[N-k] is the spectral component at frequency $\frac{N-k}{N} 2\pi$, which is equivalent to frequency $-\frac{k}{N} 2\pi$.

Derivation:

[3]Here, "distinct" means that $x_1[n] \neq x_2[n]$ for at least one value of n.

$$X^*[k] \;=\; \left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\; e^{-j\frac{2\pi}{N}kn}\right)^* \;=\; \frac{1}{N}\sum_{n=0}^{N-1} x^*[n]\; e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\; e^{j\frac{2\pi}{N}(-k)n} \qquad \text{because } x^*[n] = x[n]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\; e^{j\frac{2\pi}{N}(N-k)n} \qquad \text{because } e^{j2\pi} = 1$$

$$= X[N-k]$$

This property does not apply to complex signals.

Property D5: Conjugate pairs of coefficients synthesize a sinusoid (this is also important). When the signal $x[n]$ is real,

$$X[k]\, e^{j\frac{2\pi}{N}kn} + X[N-k]\, e^{j\frac{2\pi}{N}(N-k)n} = 2\,|X[k]|\cos\!\left(\frac{2\pi}{N}kn + \text{angle}(X[k])\right)$$

Thus, when looking at a spectrum, one should "see" cosines in the signal -- one for each conjugate pair of coefficients.

Derivation:

$$X[k]\, e^{j\frac{2\pi}{N}kn} + X[N-k]\, e^{j\frac{2\pi}{N}(N-k)n} = X[k]\, e^{j\frac{2\pi}{N}kn} + X^*[k]\, e^{j\frac{2\pi}{N}(-k)n}$$

$$= X[k]\, e^{j\frac{2\pi}{N}kn} + \left(X[k]\, e^{j\frac{2\pi}{N}kn}\right)^*$$

$$= 2\,\text{Re}\!\left(X[k]\, e^{j\frac{2\pi}{N}kn}\right)$$

$$= 2\,|X[k]|\cos\!\left(\frac{2\pi}{N}kn + \text{angle}(X[k])\right)$$

This property does not apply to complex signals.

Property D6: Linearity: Suppose $x[n]$ and $y[n]$ are periodic with period N and with $X[k]$ and $Y[k]$ as their N-point DFTs, respectively. Then the N-point DFT of $x[n] + y(t)$ is $X[k]+Y[k]$.

Similarly, if $X[k]$ and $Y[k]$ are sequences of length N, the inverse DFT of $X[k] + Y[k]$ is the sum of the inverse transforms of $X[k]$ and $Y[k]$.

The derivations of these properties are left to a homework problem.

Property D7: Parseval's theorem

For a real or complex signal $x[n]$ that is periodic with period N,

$$\text{signal power} \;=\; \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

$$= \sum_{k=0}^{N-1} |X[k]|^2$$

Recall: The power of a periodic signal $x[n]$ is

$$P(x) = \lim_{M\to\infty} \frac{1}{2M} \sum_{n=-M}^{M} |x[n]|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

Derivation:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n]\, x^*[n]$$

$$= \sum_{n=0}^{N-1} \left(\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}\right)\left(\sum_{m=0}^{N-1} X[m]\, e^{j\frac{2\pi}{N}mn}\right)^*, \quad \text{using the synthesis formula}$$

$$= \sum_{n=0}^{N-1} \left(\sum_{k=0}^{N-1} \sum_{m=0}^{N-1} X[k]\, X^*[m]\, e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}mn}\right), \quad \text{combining the two inner sums into one double sum}$$

$$= \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} X[k]\, X^*[m] \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}, \quad \text{interchanging the order of the sums}$$

$$= \sum_{k=0}^{N-1} X[k]\, X^*[k]\, N, \qquad \text{because} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n} = \begin{cases} N, & k = m \\ 0, & k \neq m \end{cases}$$

$$= N \sum_{k=0}^{N-1} |X[k]|^2$$

Though useful the remaining properties will not be emphasized in this class.

Property D8: Suppose $x[n]$ is periodic with period N, suppose $X[k]$ is the N-point DFT of $x[n]$, and suppose $X'[k]$ is the $2N$-point DFT of $x[n]$. Then,

$$X'[k] = \begin{cases} X[k/2], & k=0,2,4,... \\ 0, & k=1,1,3,... \end{cases}$$

This means that the (one-sided) spectrum based on the 2N-point DFT is the same as the one-sided spectrum based on the N-point DFT. For example if N is even, the spectrum based on the 2N-point DFT is

$$\left\{(X'[0],0),\ (X'[2],2\tfrac{2\pi}{2N}),\ ...,\ (X'[2N-2],(2N-2)\tfrac{2\pi}{2N})\right\}$$

$$= \left\{(X[0],0),\ (X[1],\tfrac{2\pi}{N}),\ ...,\ (X[N-1],(N-1)\tfrac{2\pi}{N})\right\}$$

which is the one-sided spectrum based on the N-point DFT.

The derivation is left to a homework problem.

Property D9: Time shifting: If $x[n]$ has DFT $X[k]$, then $x'[n] = x[n-n_o]$ has Fourier coefficients

$$X'[k] = X[k]\, e^{-j\frac{2\pi}{N}kn_o}$$

This shows, not surprisingly, that a time shift causes a phase shift of each spectral component, where the phase shift is proportional to the frequency of the component. The derivation is left as a homework problem.

Property D10: Frequency shifting: If x[n] has DFT X[k], then x'[n] = x[n] $e^{j\frac{2\pi}{N}k_o n}$ has Fourier coefficients

$$X'[k] = X[k-k_o].$$

This shows that multiplying a signal by a complex exponential has the effect of shifting the spectrum of the signal. The derivation is left as homework problem.

Property D11: Time scaling: This is not as straightforward as in the continuous-time case and will not be discussed here, except to indicate that if m is a positive integer, then x'[n] = x[nm] is a "subsampled" version of x[n] and x''[n] = x[n/m] is not defined for all values of n.

Property D12: Approximation by finite sums? Since the DFT synthesis formula is finite, there is no need to approximate it by a finite sum, as is often the case for Fourier series of continuous-time signal.

Property D13: Technicalities? Since the sums in the synthesis and analysis formulas are finite, no technical conditions are required as are required for the Fourier series.

E. The spectra of segments of a signal

Question: How can we assess the spectrum of a signal that is not periodic?

For example, what if the signal has finite support? Or what if the signal has infinite support, but is not periodic?

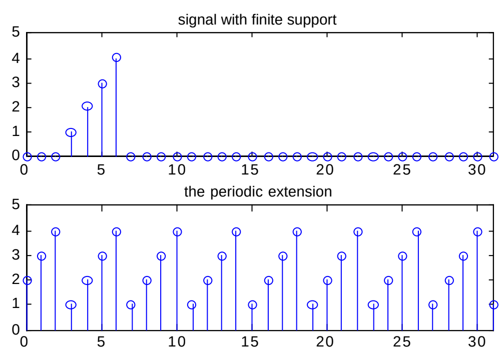
Observation: The DFT analysis formula works only with a finite segment of a signal.

Signals with finite support

We will see that to assess the spectrum of a signal $x[n]$ with finite support $[n_1,n_2]$, we can apply the DFT analysis formula directly to the signal in its support interval. Let us begin by defining $\tilde{x}[n]$ to be a periodic signal that equals $x[n]$ on the interval $[n_1,n_2]$ and simply repeats this behavior on other intervals of the same length. That is, let $N = n_2-n_1+1$

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n-mN]$$

$\tilde{x}[n]$ is called the "periodic extension" of $x[n]$. Its period N is the support length of $x[n]$. An example is given below.



Then taking the N-point DFT of $\tilde{x}[n]$ we find

$$\tilde{x}[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}, \qquad \text{(synthesis formula)}$$

where

$$X[k] = \frac{1}{N} \sum_{n=n_1}^{n_2} \tilde{x}[n]\, e^{-j\frac{2\pi}{N}kn} \qquad \text{(analysis formula)}$$

and where we have used the fact that analysis formula can limit its sum to any interval of length N. Now we note that since

$$\tilde{x}[n] = x[n] \quad \text{when } n_1 \le n \le n_2,$$

we also have

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}, \quad n_1 \le n \le n_2 \qquad \text{(synthesis formula)}$$

and

$$X[k] = \frac{1}{N} \sum_{n=n_1}^{n_2} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad \text{(analysis formula)}$$

Thus we may view the two formulas above as synthesis and analysis formulas for a spectral representation of $x[n]$. The synthesis formula shows that on its support interval, $x[n]$ can be viewed as the sum of complex exponentials with frequencies that are multiples of $2\pi/N$. The analysis formula shows how to find the spectral components. It is important to note that the synthesis formula yields $x[n]$ only in the support interval. Outside the support interval it yields $\tilde{x}[n]$, rather than $x[n] = 0$.

In summary, for a signal with finite support, we take the (one-sided) spectrum to be

$$\left\{ (X[0],0),\ (X[1],\frac{2\pi}{N}),\ (X[2],2\frac{2\pi}{N}),\ \ldots,\ (X[N-1],(N-1)\frac{2\pi}{N}) \right\},$$

just as we did for periodic signals.

Note: Though we have introduced the DFT as fundamentally applying to periodic signals and secondarily applying to signals with finite support, some people take the opposite point of view, which is also valid.

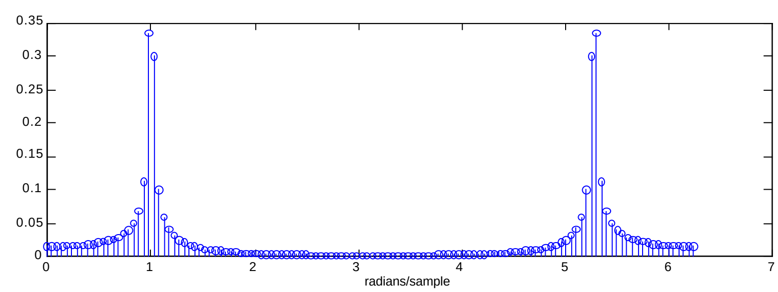
Aperiodic signals with infinite support

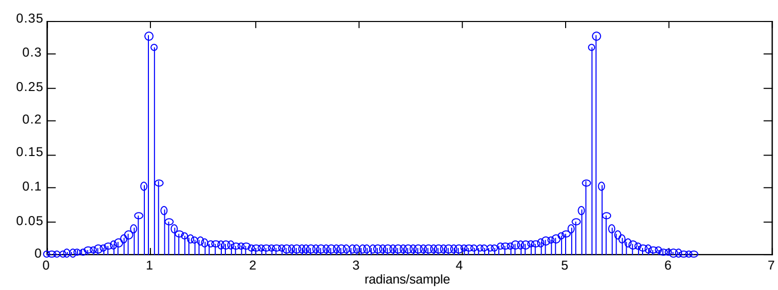
A common approach to assessing the spectrum of an aperiodic[4] signal with infinite support is to choose an integer N, divide the time axis into segments $[0,N-1]$, $[N,2N-1]$, $[2N,3N-1]$, etc, and apply the above approach to each segment. This yields a a sequence of spectra, one for each segment.

Since the signal is not periodic, the data within each segment will be different. Thus the spectrum will differ from segment to segment. For example, the spectrum of the signal

$$x[n] = \cos(3.2n)$$

is shown below for two different segments of length $N=128$.





Notice that these two spectra are quite similar. Notice also that even though the signal is a pure cosine, whose spectrum, according to the discussion of Section C, is

[4] "Aperiodic" means "not periodic".

concentrated entirely at frequencies 3.2 and 2π-3.2, the spectra above show a couple of strong components in the vicinity of 3.2 and 2π-3.2, and small components at all other harmonic frequencies. This may be viewed as being due to the fact that we are using harmonic frequencies to synthesize a sinusoid whose frequency is not harmonic. It may also be viewed as begin due to the fact that these spectra are actually the spectra of a periodic extension $\tilde{x}[n]$ of $x[n]$. A more thorough discussion, which would derive the actual form of the spectra shown above, is left to future courses.

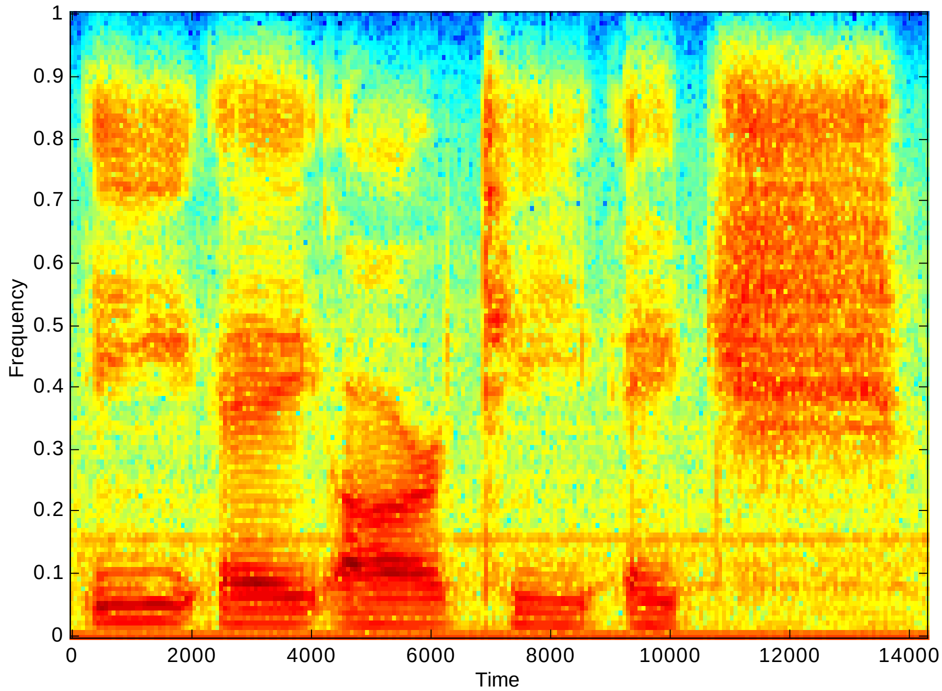
The fact that we now have two different ways of assessing the spectrum of signals such as $x[n] = \cos(3.2n)$ -- as in Section C and as discussed here -- is somewhat disconcerting. But this is reflective of the fact that, as mentioned earlier, the "spectrum" is a broad concept, like "economy" or "health", that has no simple, universal definition.

Spectrograms

Let us also mention that there are aperiodic signals for which it makes very good sense that the spectrum should differ from segment to segment. For example, the signal produced by musical instrument can be viewed as having a spectrum that changes with each note. This and other examples can be found in Section 3.5 of the text and in the Demos on the CD Rom relating to Chapter 3.

For such signals, it is common practice to apply the DFT in a sliding fashion. That is, the DFT is applied successively to overlapping intervals [0,N-1], [M,M+N-1], [2M,N+2M-1], and so on, where $M < N$. A "spectrogram" is a plot showing the magnitudes of the DFT coefficients for each interval (usually by representing the magnitude as a color or graylevel) plotted over the starting time of the interval. For example a spectrogram of someone speaking the five letters "e", "a", "r", "t", "h" is shown below[5].

[5]The colors in this plot are visible when the pdf file is displayed on a color monitor.


Frequency
Time
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
2000
4000
6000
8000
10000
12000
14000

F. The relationship between the spectrum of a continuous-time signal and that of its samples.

Frequently, we are often interested in the spectrum of some continuous-time signal, but for practical reasons, we sample the signal and work with the resulting discrete-time signal. If possible, we would like to be able to deduce the spectrum of the continuous-time signal from that of the discrete-time signal. In this section, we will show how this can be done, at least approximately.

For concreteness, consider a periodic continuous-time signal $x(t)$ with period T, and consider sampling it with sampling interval $T_S << T$. Then the resulting discrete-time signal is

$$x[n] = x(nT_S).$$

By the Fourier Series Theorem, $x(t)$ may be expressed as a sum of complex exponential components:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \, e^{j\frac{2\pi}{T}kt}$$

where

$$\alpha_k = \frac{1}{T}\int_0^T x(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$$

Can we compute or approximate the α_k's from the samples of $x(t)$, i.e. from $x[n]$? This question is answered by the following.

Fact F1: Let $x(t)$ be a periodic signal with period T, let $x[n] = x(nT_S)$ be the discrete-time signal formed by sampling $x(t)$ with sampling interval $T_S = T/N$, where $N >> 1$, and let $X[k]$ denote the N-point DFT[6] of $x[n]$.

$$\alpha_k \cong X[k] \quad \text{for } k << N$$

Derivation:

To how how α_k's can be approximated, we will use the fact that since $T_S << T$, $x(t)$ varies little over most T_S second interval. Thus, it may be approximated with

$$x(t) \cong x(nT_S) = x[n]\,, \qquad \text{when } nT_S \le t < nT_S + T_S$$

We will also use the approximation[7]

$$e^{-j\frac{2\pi}{T}kt} \cong e^{-j\frac{2\pi}{T}knT_s} = e^{-j\frac{2\pi}{N}kn}, \qquad \text{when } nT_S \le t < nT_S + T_S$$

With these approximations, we now proceed by rewriting the integral in the analysis formula as a sum of N integrals over intervals of length T_S seconds, where $N = T/T_S$:

$$\alpha_k = \frac{1}{T}\sum_{n=0}^{N-1}\int_{nT_s}^{nT_S+T_S} x(t)\, e^{-j\frac{2\pi}{T}kt}\, dt \qquad \text{integrating over short intervals}$$

---

[6]It is easy to see that $x[n]$ is periodic with period N.
[7]We will shortly discuss the validity of this approximation.

$$\cong \frac{1}{T}\sum_{n=0}^{N-1}\int_{nT_s}^{nT_s+T_s} x[n]\, e^{-j\frac{2\pi}{N}kn}\, dt \qquad \text{using the above approximations}$$

$$= \frac{1}{T}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \int_{nT_s}^{nT_s+T_s} dt \qquad \text{rearranging terms}$$

$$= \frac{1}{T}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}\, T_s \qquad \text{computing the integral}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad \text{using } N = T/T_s$$

$$= X[k] = k\text{th coefficient in N-point DFT of } x[n]$$

This fact shows that the kth Fourier coefficient α_k is approximately equal to the kth DFT coefficient $X[k]$ in an N-point DFT of $x[n]$. This means that the DFT coefficient $X[k]$ indicates the presence in $x(t)$ of the spectral component

$$X[k]\, e^{-j\frac{2\pi}{T}kt}$$

at frequency

$$\frac{2\pi}{T}k = \frac{2\pi}{N}k\frac{1}{T_s} = \frac{2\pi}{N}k\, f_s$$

where $f_s = 1/T_s$ is the sampling frequency. In other words, the component at frequency $\hat{\omega}_k = \frac{2\pi}{N}k$ in the discrete-time signal $x[n]$ represents a spectral component in the continuous-time signal $x(t)$ at frequency $\omega_k = \frac{2\pi}{N}k\, f_s$.

On the other hand, the fact that $\alpha_k \cong X[k]$ seems to contradict the facts that there are infinitely many α_k's, but only finitely many $X[k]$'s. This apparent paradox is resolved by noting that the approximation

$$e^{-j\frac{2\pi}{T}kt} \cong e^{-j\frac{2\pi}{N}kn}, \qquad \text{when } nT_s \le t < nT_s + T_s$$

is really only valid when and only when the exponential varies little within each T_s second interval. Since the exponential is periodic with period T/k, this approximation is valid when and only when $T_s \ll T/k$, or equivalently, when

$$k \ll \frac{T}{T_s} = N$$

Thus we see that the approximation $\alpha_k \cong X_k$ is really valid, only when $k \ll N$. (Actually, it turns out to be fairly good as long as $k < N/2$.)

In summary, we have shown how to approximately compute the Fourier series coefficients from samples. And the computation turns out to be the DFT!

We have also shown that the approximation is valid when $k \ll N = T/T_s$. This indicates that where possible, one should choose the sampling interval T_s to be small enough in order that the α_k's can be well approximated over whatever range of frequencies are of interest.

With this approximation for the Fourier series coefficients, one can now use the DFT coefficients to approximate the (two-sided) spectrum of the continuous-time signal $x(t)$ as

$$\left\{(X^*[K], -\tfrac{2\pi}{N}K\, f_s), \ldots, (X^*[1], -\tfrac{2\pi}{N} f_s),\ (X[0], \tfrac{2\pi}{N}K\, f_s),\ (X[1], \tfrac{2\pi}{N} f_s), \ldots, (X[K], \tfrac{2\pi}{N}K\, f_s)\right\}$$

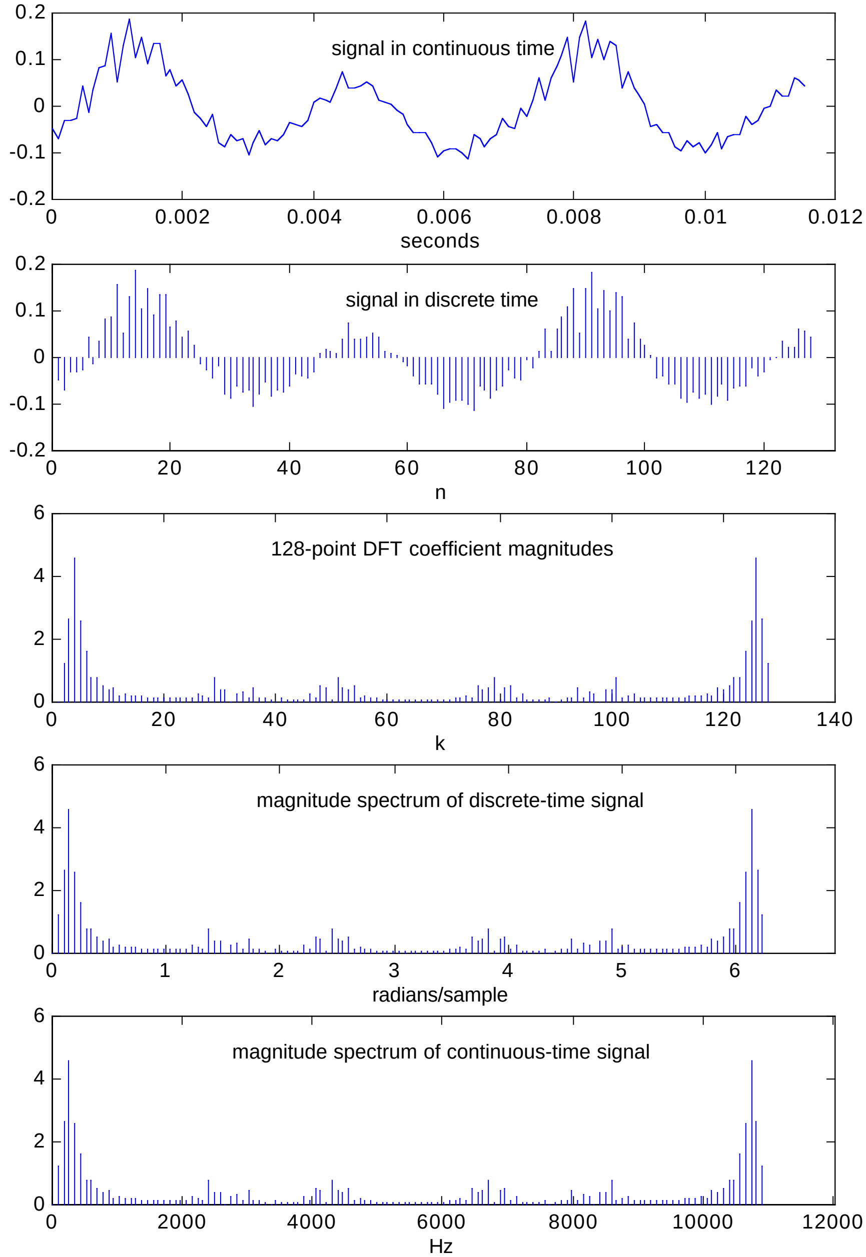
where K is the largest value of K for which we believe $\alpha_k \cong X[k]$. Note that we are, in effect, approximating the spectrum as having no components above frequency $K2\pi/T$.

Note that though we know the approximation is valid only for $k << N$, it is quite common to use the entire set of DFT coefficients in an approximation for the spectrum of $x(t)$. Specifically, it is common to plot the one-sided spectrum

$$\left\{(X[0], \tfrac{2\pi}{N}K\, f_s),\ (X[1], \tfrac{2\pi}{N} f_s), \ldots, (X[N-1], \tfrac{2\pi}{N}(N-1)\, f_s)\right\}$$

However, when interpreting such a plot, one needs to recall that such a spectrum is accurate only for values of $k << N$. Moreover, when $k > N/2$, the term $X[k] = X^*[N-k]$. Thus if anything, for $k > N/2$, the spectral line shown at frequency $\frac{2\pi}{N}(N-k)\, f_s$ is indicative of what happens at frequency $\frac{2\pi}{N} k\, f_s$ -- NOT at what happens at frequency $\frac{2\pi}{N}(N-k)\, f_s$.

Example: The figure below shows a continuous-time signal, its samples, the magnitudes of its DFT coefficients, the one-sided magnitude spectrum of the discrete-time signal, and the approximate continuous-time spectrum.



G. Bandwidth

One of the primary motivations for assessing the spectrum of signal is to find the range of frequencies occupied by the signal. This range is often called the signal's "band of spectral occupancy" or, more simply, its "band". The width of the band is called the "bandwidth". As one example, signals with nonoverlapping spectra do not interfere with each other. So if we know the band occupied by each of a set of signals, we can determine if they interfere. As another example, certain communication media, e.g. a wire, limit propagation to signals with spectral components in a certrain range. If we know the band occupied by a signal, we can determine if it will propagate.

Most signals of practical interest, such as that shown in the previous section, have spectral components extending over a broad range of frequencies. We are not really interested in the entire range of frequencies over which the spectrum is not zero. Rather we are interested in the range of frequencies over which the spectrum is "significantly large". As such, we need a definition of "significant" in order to define the concepts of "band" and "bandwidth". There are a number of such definitions in use. The definition given below is based on one such definition.

Definition:

> The "band of spectral occupancy" or "band" of a signal $s[n]$ is the smallest interval of frequencies that includes all frequencies at which the magnitude spectrum is at least one half as large as the maximum value of the magnitude spectrum.

Example:

> For the magnitude spectrum shown below, the band is, approximately, [.075, .225], and the bandwidth is .15 rad/sample.