Read Chapter 2, pp. 9–23.

1 Goals

1. Description of a sinusoid
   (a) Plotting a sinusoid
   (b) Recognizing parameters from a plot
2. Method of combining two or more sinusoids of the same frequency

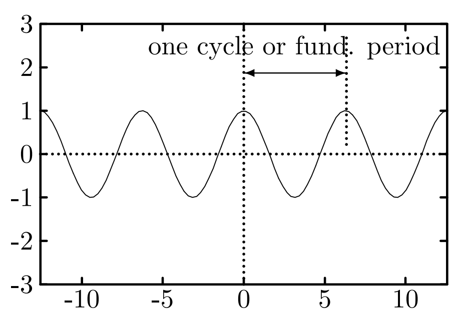
2 A Sinusoid

1. (Definition) A **sinusoid** is a signal that can be described in the following form.

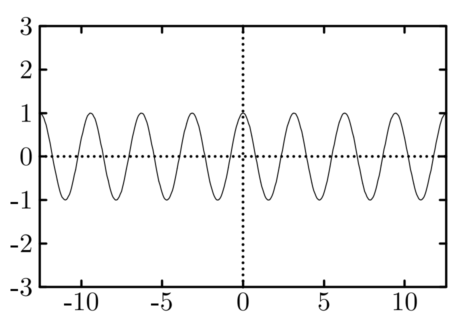
$$x(t) = A\cos(\omega_0 t + \phi), \quad A \geq 0, \quad \omega_0 \geq 0, \quad \phi \in \mathbb{R}.$$

   (a) Amplitude: A
   (b) Angular frequency or radian frequency: ω_0 [rad/s]
   (c) Phase or Phase angle: ϕ usually $[-\pi, \pi]$ or $[0, 2\pi]$

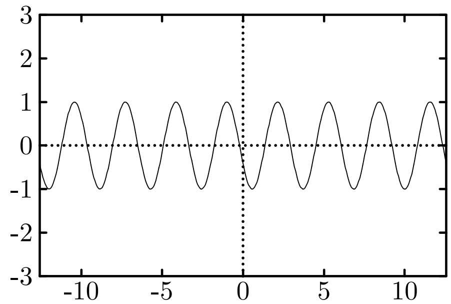
2. (Example) $x(t) = 3\cos(2t + 2)$. Convert it to $x(t) = 3\cos\big(2(t+1)\big)$
   (a) Plot $\cos(t)$: the basic form



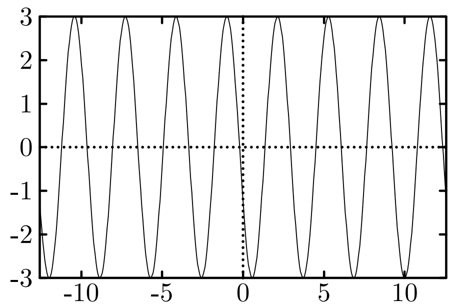
   (b) Plot $\cos(2t)$: time-scaling



(c) Plot $\cos(2\square)|_{\square=t+1}$: time-shifting by 1 to the left



(d) Magnitude Scale by 3



3. (Notes)

(a) A sinusoid is also called a *sinusoidal signal*, but rarely a cosinusoidal signal.

$$A\sin(\omega_0 t + \phi) = A\cos\left(\omega_0 t + \phi - \frac{\pi}{2}\right).$$

The choice for the standard form seems somewhat arbitrary.

i. Either choice does not lose generality: one is a phase-shifted version of the other.

ii. The cos form is preferred. One reason is that its Fourier transform is real, as opposed to complex.

(b) The amplitude of a sinusoid is nonnegative.
(Example) Find the amplitude, radian frequency, and phase angle of $x(t) = -3\cos(2t + 2)$

$$x(t) = -3\cos(2t + 2) = 3\cos(2t + 2 \pm \pi),$$

where is used a trigonometric identity $\cos(\theta \pm \pi) = -\cos\theta$.

(c) The argument of a sinusoid is angle $\omega_0 t + \phi$ measured in *radians*.

(d) The angle increases ω_0 rad/s, which is called the *angular velocity* in physics. We call it the angular frequency.

(e) A sinusoid is a periodic signal.

(f) The **fundamental period** T_0 is given by

$$T_0 = \frac{2\pi}{\omega_0}.$$

To find the fundamental period,

i. Find a positive T such that $x(t+T) = x(t)$ for all t.

ii. Find the smallest such T, if exists.

$$\begin{aligned} x(t+T) = x(t) &\implies A\cos\big(\omega_0(t+T)+\phi\big) = A\cos(\omega_0 t + \phi) \\ &\implies \omega_0 T = 2\pi n, \quad n = 1, 2, \dots. \end{aligned}$$

Then

$$T = \frac{2\pi n}{\omega_0}.$$

4. Two forms of frequency

   (a) (Frequency) Instead of the angular (radian) frequency, we often use *frequency* as the number of periods per second.

   $$\frac{1}{T_0} = \frac{\text{no. of fundamental periods}}{\text{second}} = \frac{\text{no. of cycles}}{\text{second}} = f_0$$

   The unit is cycles/s or Hertz [Hz].

   (b) (Frequency Relationship)

   $$\begin{aligned} T_0 = \frac{2\pi}{\omega_0}, \quad f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi} \\ \implies f_0 = \omega_0 2\pi \quad \text{or} \quad \omega_0 = 2\pi f_0. \end{aligned}$$

5. (Phase angle or phase shift) The phase angle ϕ is equivalent to *time-shift* by ϕ/ω_0 to the left.

   $$\begin{aligned} x(t) = A\cos(\omega_0 t + \phi), \quad y(t) = A\cos(\omega_0 t) \\ \implies \quad x(t) = y\big(t + \frac{\phi}{\omega_0}\big) = A\cos\big(\omega_0(t + \frac{\phi}{\omega_0})\big). \end{aligned}$$

6. (Zero Frequency) For $\omega_0 = 0$ or $f_0 = 0$

   $$x(t) = A\cos(2\pi f_0 t + \phi) = A\cos\phi.$$

   A constant (dc) signal is a degenerated case of a sinusoid.

7. (Negative Frequency for Sinusoids?)

   $$\begin{aligned} \cos(-2\pi f_0 t) = \cos(2\pi f_0 t) \\ \cos(-2\pi f_0 t + \phi) = \cos(2\pi f_0 t - \phi) \end{aligned}$$

   We require that f_0 be nonnegative by convention. We do *not* lose generality in so doing.

3 Parameter Identification

Statement Given a plot of a sinusoid

$$x(t) = A\cos(\omega_0 t + \phi) = A\cos\Big(\omega_0\big(t + \frac{\phi}{\omega_0}\big)\Big),$$

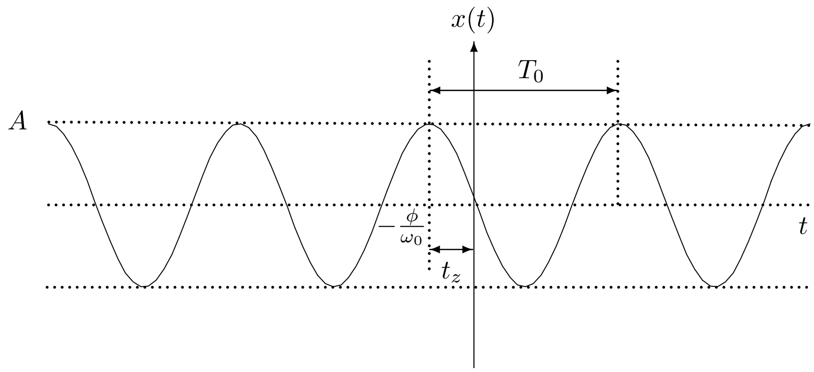
identify A, ω_0 and ϕ.

1. $A = \max_t x(t)$

2. $\omega_0 = \frac{2\pi}{T_0}$, where T_0 is the fundamental period, which equals the time span for one cycle.

3. The "first" maximum occurs at $t = -\frac{\phi}{\omega_0}$. Measure t_z and then

$$t_z = \frac{\phi}{\omega_0} \qquad \Longrightarrow \phi = \omega_0 t_z.$$



4 Sums of Sinusoids

We will develop systematic methods to find and simplify a sum of sinusoids using complex exponential signals.

1. (Fact) A sum of multiple sinusoids of the **same frequency** is a sinusoid of that frequency.

$$A\cos(\omega_0 t + \phi) + B\cos(\omega_0 t + \psi) = C\cos(\omega_0 t + \theta)$$

2. (Fact) A sum of two sinusoids of the different frequencies is *not* a sinusoid.

   (a) If the ratio of the periods is rational, then it is periodic.

   (b) Otherwise, it is aperiodic.

3. We will develop a method, where we utilize

$$\text{sinusoid} \quad = \quad \text{Real Part of a complex exponential signal.}$$