1 Goals

1. Spectra of sinusoidal-sum signals
2. Motivation for studying spectra
3. How does one assess the spectrum of a given signal?
   (i) a signal that is a sum of sinusoids
   (ii) a periodic signal via Fourier series
   (iii) segments of a signal

2 Spectrum of Continuous-Time Signals

2.1 Spectra of Sinusoidal-Sum Signals

Suppose that a signal $x(t)$ is a sum of sinusoids, i.e.,

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k),$$

where $A_k \geq 0$ for $k = 1, 2, \ldots$.

- (**Single-Sided Spectrum**) The single-sided spectrum is the list (collection) of pairs $(A_k \angle \phi_k, f_k)$ for $k = 0, 1, 2, \ldots$. That is, the spectrum of $x(t)$ is

$$\Big\{(A_0, 0), (A_1 \angle \phi_1, f_1), \ldots, (A_N \angle \phi_N, f_N)\Big\}.$$

  The terms $A_k \cos(2\pi f_k t + \phi_k)$ are called the sinusoidal components.

- (**Two-Sided Spectrum**) The two-sided spectrum of $x(t)$ is the list (collection) of pairs (C_k, f_k) for $k = 0, \pm 1, \pm 2, \ldots$, where

$$C_0 = A_0,$$
$$C_k = \begin{cases} \frac{A_k e^{j\phi_k}}{2}, & k > 0, \\ \frac{A_k e^{-j\phi_k}}{2}, & k < 0. \end{cases}$$

  That is, the spectrum of $x(t)$ is

$$\Big\{(C_0, 0), (C_1, f_1), (C_{-1}, -f_1), \ldots, (C_N, f_N), (C_{-N}, -f_N)\Big\}.$$

  (a) The spectrum of a signal is also called the **frequency-domain representation** (or spectral representation) of the signal. In contrast, $x(t)$ as a function of time is called the **time-domain representation** of the signal.

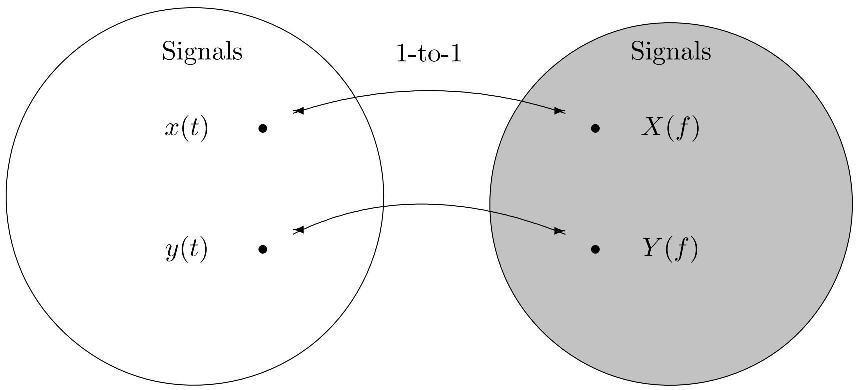
  (b) The terms $C_k e^{j2\pi f_k t}$ are called the **complex exponential components** or spectral components of the signal.

Then $x(t)$ is expressed in terms of (C_k, f_k) by

$$x(t) = \sum_{k=-N}^{N} C_k e^{j2\pi f_k t}.$$

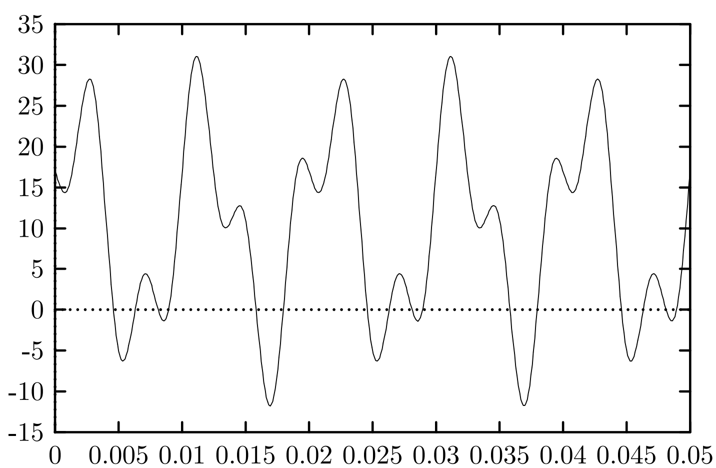
The spectrum $X(f)$ as a function of frequency then is given by

$$X(f) = \begin{cases} C_k, & f = f_k, \\ 0, & \text{otherwise.} \end{cases}$$



Example 2.1 Let

$$x(t) = 10 + 14\cos(2\pi 100t - \frac{\pi}{3}) + 8\cos(2\pi 250t + \frac{\pi}{2}).$$



(a) Find the (two-sided) spectrum of $x(t)$.

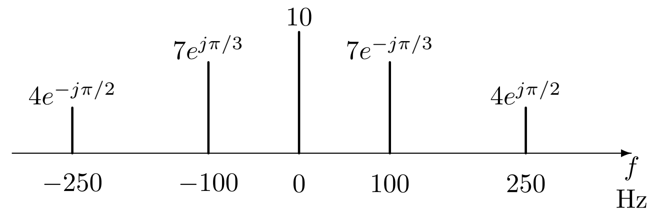
Using the inverse Euler formula,

$$\begin{aligned} x(t) &= 10 + 14 \cdot \frac{e^{j(2\pi 100t - \pi/3)} + e^{-j(2\pi 100t - \pi/3)}}{2} + 8 \cdot \frac{e^{j(2\pi 250t + \pi/2)} + e^{-j(2\pi 250t + \pi/2)}}{2} \\ &= 10 + 7e^{-j\pi/3}e^{j2\pi 100t} + 7e^{j\pi/3}e^{-j2\pi 100t} + 4e^{j\pi/2}e^{j2\pi 250t} + 4e^{-j\pi/2}e^{-j2\pi 250t}. \end{aligned}$$

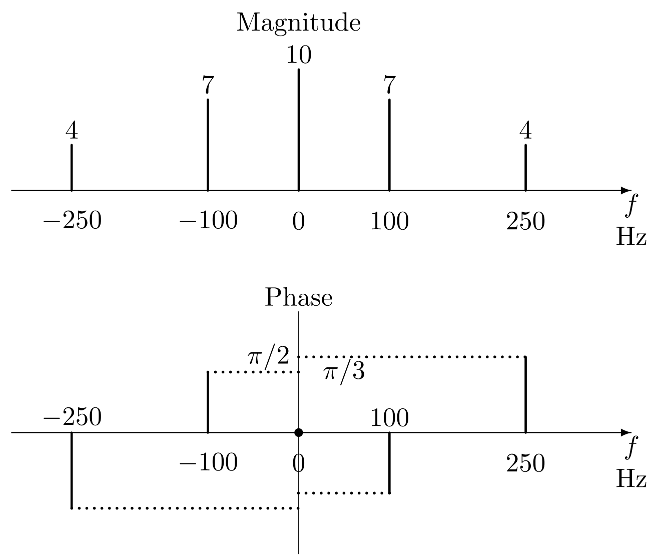
Then the spectrum of $x(t)$ is

$$\left\{(10,0),(7e^{-j\pi/3},100),(7e^{j\pi/3},-100),(4e^{j\pi/2},250),(4e^{-j\pi/2},-250)\right\}$$

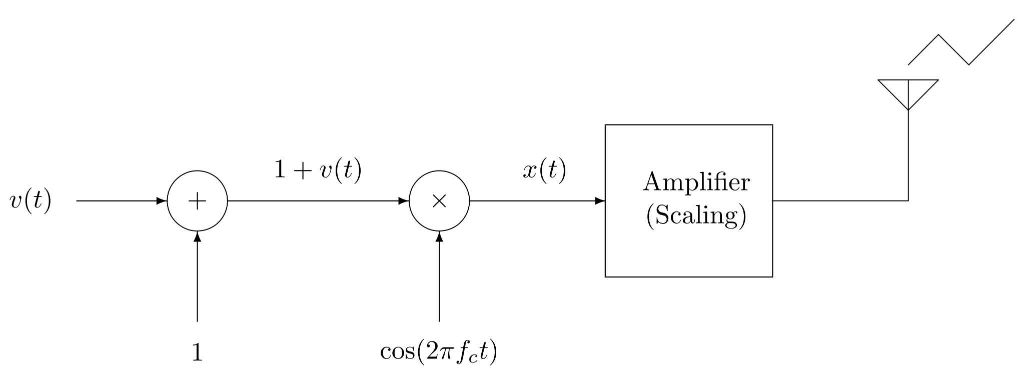
(b) Plot the spectrum.



Alternatively,



Example 2.2 AM broadcasting uses a signal $x(t)$ generated as shown in the diagram.



where

(i) $v(t)$ is the audio (message) signal with $|v(t)| \leq 1$ for all t,

(ii) $\cos(2\pi f_c t)$ is the carrier (signal), where the carrier frequency f_c ranges from 540 to 1,600 kHz, to which we tune the radio.

Motivation for modulation

- An audio signal is low frequency, typically 0 to 5 kHz.
- Low frequencies do not radiate and propagate well through the atmosphere.
- We need to generate a high frequency signal that will *carry* the audio signal.
- $\cos(2\pi f_c t)$ is the carrier signal. It has high frequency.
- $x(t)$ is obtained by *modulating* the carrier by the audio signal, specifically

$$x(t) = \underbrace{\big(1 + v(t)\big)}_{\text{envelope}} \cos(2\pi f_c t).$$

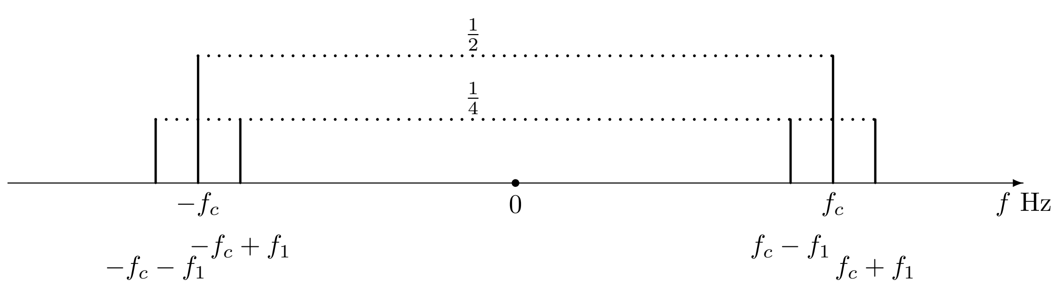
(a) Find the spectrum of $x(t)$ for $v(t) = \cos(2\pi f_1 t)$.

Note that

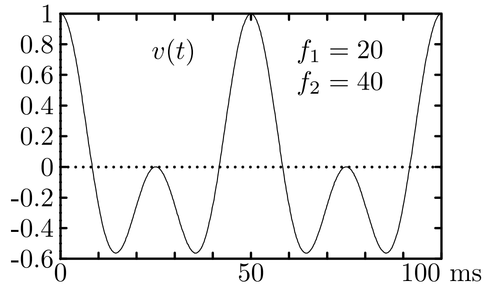
$$\begin{aligned} x(t) &= \cos(2\pi f_c t) + \cos(2\pi f_1 t)\cos(2\pi f_c t) \\ &= \cos(2\pi f_c t) + \frac{\cos\big(2\pi(f_c+f_1)t\big) + \cos\big(2\pi(f_c-f_1)t\big)}{2} \qquad \because \text{prod-to-sum formula} \\ &= \cos(2\pi f_c t) + \frac{1}{2}\cos\big(2\pi(f_c+f_1)t\big) + \frac{1}{2}\cos\big(2\pi(f_c-f_1)t\big). \end{aligned}$$

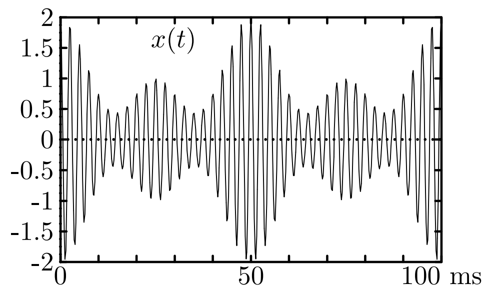
Then by inspection the spectrum of $x(t)$ is

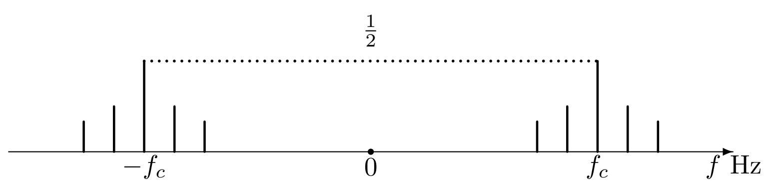
$$\Big\{(\tfrac{1}{2}, f_c), (\tfrac{1}{2}, -f_c), (\tfrac{1}{4}, f_c+f_1), (\tfrac{1}{4}, f_c-f_1), (\tfrac{1}{4}, -f_c-f_1), (\tfrac{1}{4}, -f_c+f_1)\Big\}$$



(b) Find the spectrum of $x(t)$ for $v(t) = 0.5\cos(2\pi f_1 t) + 0.5\cos(2\pi f_2 t)$. Assume $f_2 > f_1$.



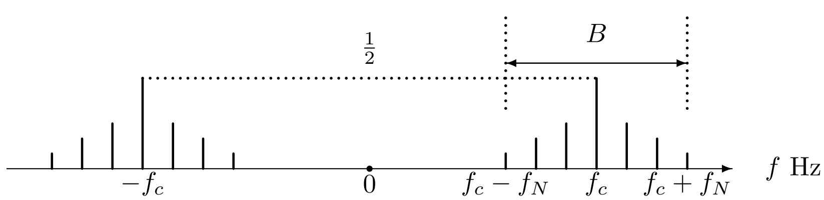




(c) Find the spectrum of $x(t)$ for

$$v(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t).$$

Assume that A_k are chosen such that $|v(t)| \leq 1$ for all t and $f_{k+1} > f_k$ for all k.
By inspection $\frac{1}{2}$ at $f = \pm f_c$ and $\frac{A_k}{4}$ at $f = \pm f_k$ for $k = 1, 2, \ldots, N$.



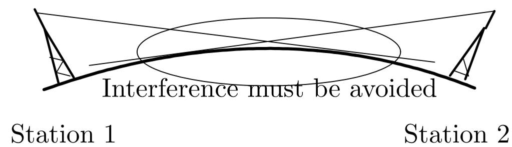
The frequency range $[f_c - f_N, f_c + f_N]$ must be within the specification by the Federal Frequency Commission. Currently, it is $[f_c - 5, f_c + 5]$ kHz.

Example 2.3 Frequency multiplexing of AM signals uses spectra to design a working system.
Suppose that you are an FCC officer to allocate frequencies to prospective AM stations in the following situation.

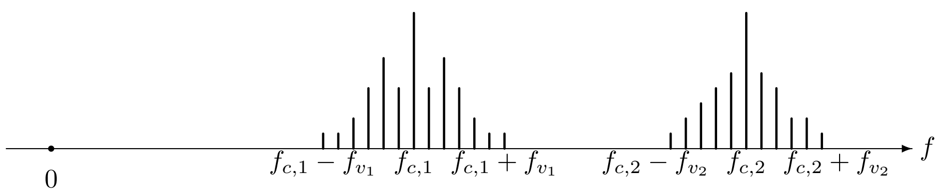
- Station 1 wants to broadcast audio signal $v_1(t)$ at carrier frequency $f_{c,1}$.
- Station 2 wants to broadcast audio signal $v_2(t)$ at carrier frequency $f_{c,2}$. Assume $f_{c,2} > f_{c,1}$.

Audio signals have frequency components up to 5 kHz.

Task Find out how far apart $f_{c,1}$ and $f_{c,2}$ must be so that the two transmitted signals may not interfere with each other.



Solution Let the maximum audio frequency components are at $f = f_{v_1}$ and f_{v_2} for $v_1(t)$ and $v_2(t)$, respectively. (Note that $f_{v_1}, f_{v_2} \leq 5$ kHz.



By inspection, no interference requires that $f_{c,2} - f_{v_2} > f_{c,1} + f_{v_1}$, which implies that

$$f_{c,2} - f_{c_1,1} > f_{v_1} + f_{v_2} \geq 10 \text{ kHz}.$$

□

2.2 Why in terms of Sinusoids or Complex Exponentials?

When we discuss spectra, we always use sinusoids and/or complex exponential signals for spectral components. Why is that?

(a) (SISO for Linear Systems) It is mainly that sinusoids into linear systems lead to sinusoids. (No other class of signals has this property.)

(b) This causes the input-output relationship for linear systems to be particularly simple for sinusoidal signals.

(c) So representing signals with sinusoids simplifies analysis greatly.

(d) Because analysis is simplified, efficient design methods can be developed.

3 Motivation for Studying Spectrum

A signal and its spectrum are like the two sides of a coin. So a signal can be described either as a function of time or as a function of frequency. Mathematically, they have the same information. Then why are we interested in spectrum? Isn't it true that a signal as a time function would suffice, since the function representation contains all the information there is?

We are interested in spectral representation of a signal because

(a) sometimes it is simpler than the time-domain representation,

(b) sometimes it is more intuitively informative than the time-domain representation,

(c) sometime it is better understood in the frequency domain,

(d) sometime it better captures a physical phenomenon in the frequency domain.

Some of the situations when spectrum proves better include the following.

- (Multiplexing) Signals with nonoverlapping spectrum do not interfere with one another. Thus many information carrying signals can be transmitted over a single communication medium (wire, fiber, cable, atmosphere, water, etc.). To design such systems, we need to be able to quantitatively determine the spectrum of signals to be able to assess whether or not they overlap, and if they do, by how much. Also, we need to able to develop systems (e.g. filters) that select one signal over another, based on its spectrum.

- (Unique Spectral Features) Some signals can be recognized based on their spectra, e.g. vowels (Labs 8,9), touchtone telephone key presses, musical notes and chords, bird songs, whale sounds, mechanical vibration analysis, atomic/molecular makeup of sun and other stars, etc. To build systems that automatically recognize such signals, we need to able to quantitatively determine the spectrum of a signal.

- (Propagation Characteristics) Communication media, e.g. the atmosphere, the ocean, a wire, an optical fiber, often limit propagation to signals with components only in a certain frequency range (atmosphere is high frequency, ocean is low frequency, wire is low frequency, optical fiber is high frequency, but what is considered "high" or "low" depends on the media. We need to be able to assess the spectrum of a signal to see if it will propagate. We need to be able to design signals to have appropriate spectra for appropriate media.

- (Signal Analysis) In many situations, the behavior of many natural and man-made linear systems is best analyzed in the "frequency domain", i.e. one determines the behavior in response to sinusoids (or complex exponentials) at various frequencies, and from this one can deduce the response to other signals. The previous bullet is a special case of this.

- (Filtering) In many situations, an undesired signal interferes with a desired signal, e.g. the desired signal might correspond to someone speaking and the undesired signal might be background noise. We wish to reduce or eliminate the background signal. In order to be able to reduce or eliminate the background signal it must have some characteristic that is distinctly different than the desired signal. Often it happens that the desired and undesired signals have distinctly different spectra (e.g. the noise has mostly high frequency components). In such cases, one can design systems, called "filters", that selectively reduce certain frequency components. These can be used to reduce the noise while having little effect on the desired signal.

- (Frequency Sensitive Systems) Many other signals and systems methods are based on spectra: e.g. control engineering, data compression, voice recognition, music processing.

4 How to Find Spectrum

(a) Sinusoidal-sum signals: the inverse Euler formulas

(b) Periodic signals: the Fourier series

(c) Segments of a signal: the Fourier series