# Goal: How to Find Spectrum

1. Sinusoidal-sum signals: the inverse Euler formulas
2. Periodic signals: the Fourier series
3. Segments of a signal: the Fourier series

# 1 Spectrum of Continuous-Time Signals

## 1.1 Spectra of Sinusoidal-Sum Signals

Suppose that a signal $x(t)$ is a sum of sinusoids, i.e.,

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k),$$

where $A_k \geq 0$ for $k = 1, 2, \ldots$.

Then the spectrum is the collection $\{(C_k, f_k)\}_{k=-N}^{N}$ with $f_{-k} = -f_k$, where

$$C_0 = A_0,$$
$$C_k = \begin{cases} \frac{A_k e^{j\phi_k}}{2}, & k > 0, \\ \frac{A_k e^{-j\phi_k}}{2}, & k < 0. \end{cases}$$

Conversely, given the spectrum $\{(C_k, f_k)\}_{k=-N}^{N}$ with $f_{-k} = -f_k$, the signal is specified as follows:

$$x(t) = \sum_{k=-N}^{N} C_k e^{j2\pi f_k t}, \quad f_{-k} = -f_k.$$

It shows that $x(t)$ contains $C_k$ of complex exponential component $e^{j2\pi f_k t}$.

## 1.2 Spectrum of a Periodic Signal

The main point of this section is the following theorem, which we won't prove, but which we will illustrate and use.

**Theorem 1.1 Fourier Series Theorem**

A periodic signal $x(t)$ with period $T_0$ can be written as an infinite sum of sinusoids, all of which have frequencies that are multiples of $1/T_0$. That is, there are a set of amplitudes and phases $(A_0, \phi_0)$, $(A_1, \phi_1)$, $(A_2, \phi_2)$, …, such that

- $$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi \frac{k}{T_0} t + \phi_k\right). \quad (1)$$

- Or equivalently,

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Big( \frac{X_k}{2} e^{j2\pi \frac{k}{T_0} t} + \frac{X_k^*}{2} e^{j2\pi \frac{k}{T_0} t} \Big), \tag{2}$$

where

$$X_0 = A_0, \quad X_k = A_k e^{j\phi_k}.$$

- Or equivalently,

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi \frac{k}{T_0} t}, \tag{3}$$

where

$$C_0 = X_0 = A_0, \quad C_k = \begin{cases} \frac{A_k e^{j\phi_k}}{2}, & k \geq 1, \\ \frac{A_k e^{-j\phi_k}}{2}, & k \leq -1. \end{cases}$$

**Appreciation**

(a) (Periodic signal as a sinusoidal sum)

(i) It says that *any* periodic signal can be represented as a *sum of sinusoids.* (Well, not all, but any practical periodic signals.)

(ii) But it may take an infinite number of them.

(iii) The term $\cos\left(2\pi \frac{k}{T_0} t + \phi_k\right)$ is the "sinusoidal component" of $x(t)$ at frequency $\frac{k}{T_0}$ Hz.

(iv) Note that all sinusoids in the above have frequencies that are multiples of $\frac{k}{T_0}$ Hz.

(b) (Periodic signal as a complex-exponential sum)

(i) It also says that *any* periodic signal can be represented as a *sum of complex exponentials.* (It may take an infinite number.)

(ii) The term $C_k e^{j2\pi \frac{k}{T_0} t}$ is the "complex exponential component" of $x(t)$ at frequency $\frac{k}{T_0}$ Hz.

(c) It follows from this theorem that the spectrum of a periodic signal with period $T_0$ is concentrated at frequencies

$$0, \pm\frac{1}{T_0}, \pm\frac{2}{T_0}, \ldots,$$

or some subset thereof, i.e. $x(t)$ has spectral components only at these frequencies.

(i) (Fundamental frequency) $1/T_0$

(ii) (Harmonic frequencies) multiples of the fundamental frequency.

(d) (Fourier Series) The three sums given above are considered to be three forms of the "Fourier series." (A *series* is an infinite sum.)

(i) The book introduces the first two forms in Section 3.4 equation (3.4.1).

(ii) We'll use the third form. It is most common to use the third form, because it is easier to work with.

(iii) The $A_k$'s, $\phi_k$'s, $X_k$'s and $C_k$'s are called *Fourier series coefficients* or just *Fourier coefficients.*

(e) (Spectrum) To "find the spectrum of a periodic signal" we need to find $T_0$ and the Fourier coefficients $C_k$'s. Here's the formula:

$$C_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi\frac{k}{T_0}t}\,dt = \frac{1}{T_0}\int_{T_0} x(t)e^{-j2\pi\frac{k}{T_0}t}\,dt.$$

(i) The derivation is well presented in the new section 3.4.5.

(ii) Notice that $C_k$ is the correlation of $x(t)$ with $e^{j2\pi\frac{k}{T_0}t}$ normalized by $1/T_0$, which is the energy of one period of the exponential.

(iii) The term $C_k e^{j2\pi\frac{k}{T_0}t}$ is the component of $x(t)$ that is similar to $e^{j2\pi\frac{k}{T_0}t}$.

(iv) The Fourier coefficient $C_k$ measures the degree of similarity of $x(t)$ to the exponential.

(v) There's a similar interpretation that $A_k\cos\left(2\pi\frac{k}{T_0}t + \phi_k\right)$ is the component of $x(t)$ that is like a cosine at frequency $\frac{k}{T_0}$ Hz.

(f) Summing of sinusoids can generate an arbitrary signal

(i) Demo program from Lab 3 "sinsum".

(ii) Matlab demo program called `xfourier.m`.

### 1.2.1 Summary of Fourier series

Synthesis Formula: shows how $x(t)$ is a sum of complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi\frac{k}{T_0}t}.$$

Analysis Formula: shows how to compute the $C_k$'s, i.e. the Fourier coefficients

$$C_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi\frac{k}{T}t}\,dt.$$

Thus, finding the spectrum of a periodic signal involves finding the period and the $C_k$'s. Finding the $C_k$'s is often called "taking the Fourier series."

- (One-to-oneness) There is a one-to-one relationship between periodic signals with period $T_0$ and sequences of Fourier coefficients. In the Fourier series theorem, it can be shown that there is one and only one set of coefficients that works in the synthesis formula, i.e. There is one and only one set of coefficients $\{C_k\}_{k=-\infty}^{\infty}$ such that

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi\frac{k}{T_0}t}.$$

Thus the following two statements are valid.

(a) If you find a set of coefficients such that $x(t) = \sum_k C_k e^{j2\pi\frac{k}{T_0}t}$, then these are necessarily the Fourier coefficients.

(b) And

$$\sum_{k=-\infty}^{\infty} C_k e^{j2\pi\frac{k}{T_0}t} = \sum_{n=-\infty}^{\infty} D_n e^{j2\pi\frac{n}{T_0}t} \implies \quad C_k = D_k \quad \forall k.$$

This means that in some cases, the Fourier coefficients can be found by inspection.

Equivalently,

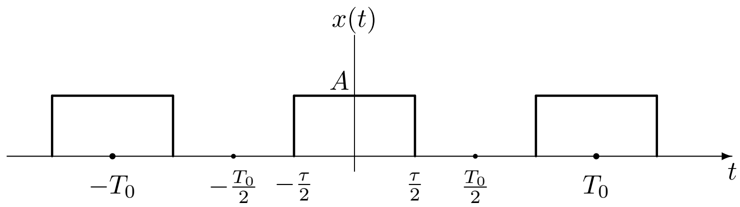
(c) If $x_1(t)$ and $x_2(t)$ are different signals in the sense that their difference has nonzero power, each with period $T_0$, then

$$C_k \text{ for } x_1(t) \quad \neq \quad C_k \text{ for } x_2(t) \quad \text{for at least one } k.$$

- (Choice of period) If a signal has period $T$, then it also has period $2T$. So when applying Fourier analysis we have a choice as to $T$. Often, but certainly not always, we choose $T$ to equal the fundamental period $T_0$. If we want to specify the value of $T$, we say the "$T$-second Fourier series".
- (Other forms) If you wish to find the other forms of the Fourier series, use the formulas:

$$A_0 = C_0,$$
$$A_k = 2|C_k|, \quad \phi_k = \angle C_k, \quad k = 1, 2, \ldots.$$

**Example 1.1** [Spectrum of a periodic signal] Find the spectrum of the following periodic signal with period $T_0$.




Let's find the complex Fourier series representation of $x(t)$.

(a) Find the (fundamental period) $T_0$. Well, it's $T_0$.

(b) Find $C_0$.

$$C_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{A\tau}{T_0}.$$

(c) Find $C_k$ for $k \neq 0$.

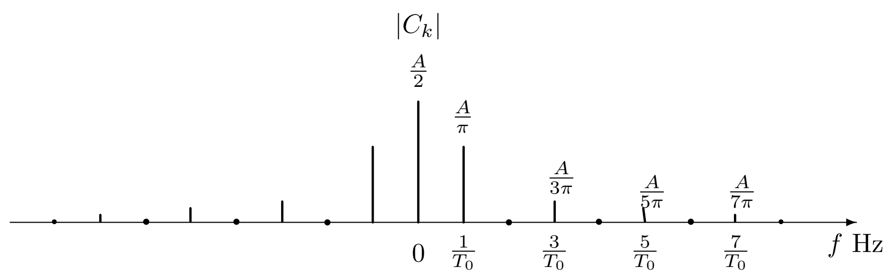
$$\begin{aligned}
C_k &= \frac{1}{T_0}\int_{T_0} x(t)e^{-j2\pi kt/T_0}\,dt \\
&= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} Ae^{-j2\pi kt/T_0}\,dt \\
&= \frac{A}{T_0}\int_{-\tau/2}^{\tau/2} e^{-j2\pi kt/T_0}\,dt \\
&= \frac{A}{T_0}\cdot\frac{1}{-j2\pi k/T_0}e^{-j2\pi kt/T_0}\Big|_{-\tau/2}^{\tau/2} \\
&= \frac{A}{T_0}\cdot\frac{1}{-j2\pi k/T_0}\left(e^{-j\pi k\tau/T_0} - e^{-j\pi k\tau/T_0}\right) \\
&= \frac{A}{T_0}\cdot\frac{1}{\pi k/T_0}\left(\frac{e^{j\pi k\tau/T_0} - e^{-j\pi k\tau/T_0}}{j2}\right) \\
&= \frac{A}{\pi k}\sin\left(\pi k\tau/T_0\right)
\end{aligned}$$

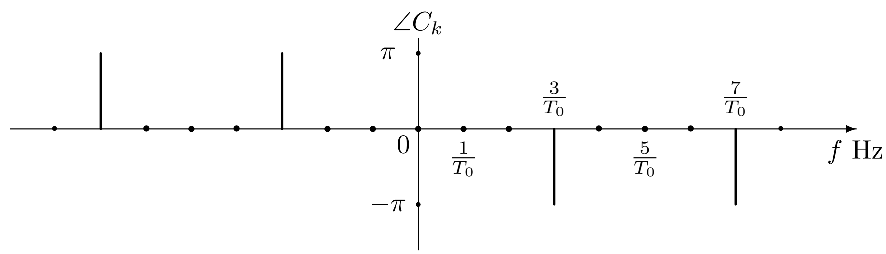
(d) Then

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T_0}$$
$$= \frac{A\tau}{T_0} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \underbrace{\frac{A}{\pi k}\sin\left(\pi k\tau/T_0\right)}_{\text{coefficient}} e^{j2\pi kt/T_0}.$$

(e) (Special case) Let $\tau = T_0/2$. Then

$$C_0 = \frac{A}{2},$$
$$C_k = \frac{A}{\pi k}\sin\left(\frac{\pi k}{2}\right) = \begin{cases} \frac{A}{\pi k}(-1)^{(k-1)/2}, & k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$




Several terms of the Fourier series of $x(t)$ reveal that

$$x(t) = \frac{A}{2} + \frac{A}{\pi}e^{j2\pi t/T_0} + \frac{A}{\pi}e^{j2\pi t/T_0}$$
$$- \frac{A}{3\pi}e^{j2\pi 3t/T_0} - \frac{A}{3\pi}e^{j2\pi 3t/T_0} + - \cdots$$
$$= \frac{A}{2} + \frac{2A}{\pi}\cos(2\pi\frac{1}{T_0}t) - \frac{2A}{3\pi}\cos(2\pi\frac{3}{T_0}t) + - \cdots$$
$$= \frac{A}{2} + \frac{2A}{\pi}\cos(2\pi\frac{1}{T_0}t) + \frac{2A}{3\pi}\cos\left(2\pi\frac{3}{T_0}t - \pi\right) + \cdots.$$

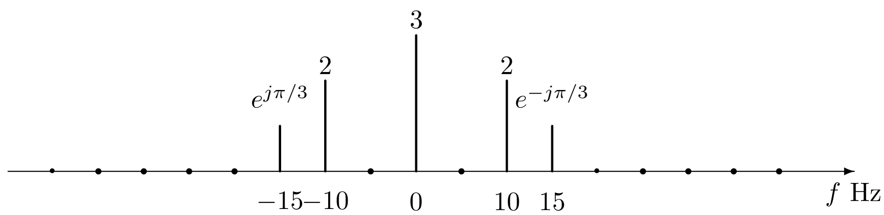
**Example 1.2** [Spectrum of a periodic signal that is a finite sum of sinusoids]
Find the spectrum of the following signal.

$$x(t) = 3 + 4\cos(2\pi 10t) + 2\cos\left(2\pi 15t - \frac{\pi}{3}\right).$$

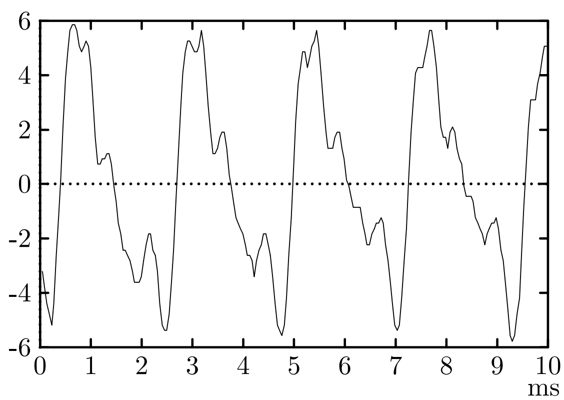
Note that $x(t)$ is periodic with fundamental period $T_0 = \frac{1}{5}$. So, we can find its spectrum using the Fourier series theorem. Or we can find it by inspection just as we did for finite sums of sinusoids. The one-to-oneness of the relation between Fourier coefficients and periodic signals means that the coefficients we obtain by inspection are the Fourier series coefficients.

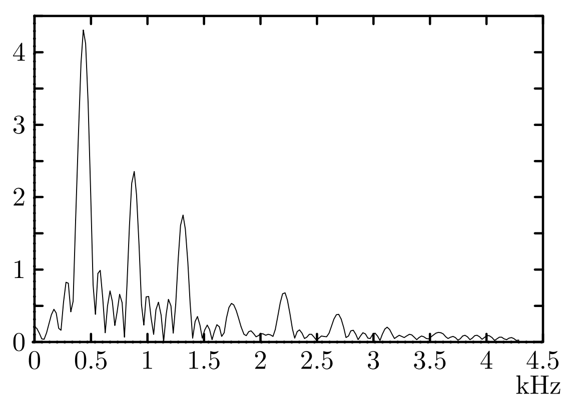
By inspection (or the inverse Euler formula),

$$x(t) = 3 + 2e^{j2\pi 10t} + 2e^{-j2\pi 10t} + e^{-j\pi/3}e^{j2\pi 15t} + e^{j\pi/3}e^{-j2\pi 15t}.$$




**Example 1.3** The plot below shows a segment of note A above the middle C of the piano. Its period is about $T_0 = 2.27$ ms. Its spectral lines are located at 440 Hz and its multiples. Note that 440 Hz is the reciprocal of 2.27 ms.



### 1.2.2 More Properties of Fourier Series Coefficients

- (DC/average/mean value) $\text{MV}(x) = C_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$.
- (Integration over one period)

$$\begin{aligned} C_k &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}\,dt \\ &= \frac{1}{T_0}\int_{\tau}^{\tau+T_0} x(t)e^{-j2\pi kt/T_0}\,dt \quad \text{any } \tau \\ &= \frac{1}{T_0}\int_{T_0} x(t)e^{-j2\pi kt/T_0}\,dt. \end{aligned}$$

- (Conjugate symmetry)
  (a) For real signal $x(t)$

$$C_{-k} = C_k^*.$$

  (b) Conjugate components of real $x(t)$ synthesize a sinusoids

$$C_k e^{j2\pi kt/T_0} + C_{-k}e^{-j2\pi kt/T_0} = 2|C_k|\cos\left(2\pi\frac{k}{T_0}t + \angle C_k\right).$$

  Thus when looking at a spectrum one "sees" cosines—one for every conjugate pair of coefficients.

- Parseval's theorem

$$\text{signal power} = \text{MSV}(x) = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |C_k|^2.$$

- Approximating the Fourier synthesis formula with a finite number of terms, i.e.

$$x(t) \approx \sum_{k=-N}^{N} C_k e^{j2\pi kt/T_0}.$$

  This is necessary in many practical cases. It can be shown that the difference signal

$$e(t) = x(t) - \sum_{k=-N}^{N} C_k e^{j2\pi kt/T_0}$$

  has power $2\sum_{k=N+1}^{\infty}|C_k|^2$, which goes to zero as $N$ increases. So choose $N$ so large that this is small.

- Mathematical Technicalities: Need to assume $\int_{T_0}|x(t)|dt < \infty$ and/or $\int_{T_0}|x(t)|^2dt < \infty$.

  When mathematicians prove $x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi\frac{k}{T_0}t}$, what they really show is the power in the difference signal $e(t) = x(t) - \sum_{k=-\infty}^{\infty} C_k e^{j2\pi\frac{k}{T_0}t}$ is zero, assuming $\int_{T_0}|x(t)|^2dt < \infty$. So $x(t)$ and $\sum_{k=-\infty}^{\infty} C_k e^{j2\pi\frac{k}{T_0}t}$. Moreover, assuming $\int_{T_0}|x(t)|dt < \infty$ and "Dirichlet conditions", the only points at which they can differ are points of discontinuity in $x(t)$. Specifically

  (a) $\sum_{k=-\infty}^{\infty} C_k e^{j2\pi\frac{k}{T_0}t} = x(t)$ if $t$ is a point of continuity, and

  (b) $\sum_{k=-\infty}^{\infty} C_k e^{j2\pi\frac{k}{T_0}t} = \frac{x(t^-)+x(t^+)}{2}$ if there is a discontinuity at $t$.

- The Gibbs phenomenon: the "overshooting" at discontinuities never goes away completely.

## 1.3 Spectra of Segments of a Signal

**Motivating question** What is the spectrum of a signal that is not periodic? For example,

(a) what if the signal has finite support?

(b) Or what if the signal has infinite support, but is not periodic?

**Observation** The Fourier series analysis formula works only with a *finite segment* of a signal.

**Approach for finite support** If the signal has finite support $[t_1, t_2]$, apply the analysis formula

$$C_k = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x(t) e^{j2\pi kt/(t_2 - t_1)} \, dt.$$

Now let

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi \frac{k}{T} t}, \quad \text{where } T = t_2 - t_1.$$

Then $\tilde{x}(t)$ is periodic with period $T$, and

$$\tilde{x}(t) = x(t), \quad \text{when } t_1 \leq t \leq t_2.$$

Therefore,

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi \frac{k}{T_0} t}, \quad \text{when } t_1 \leq t \leq t_2,$$

which is a synthesis formula for $x(t)$ that works only for the support interval $t_1 \leq t \leq t_2$.
$\tilde{x}(t)$ is called the *periodic extension* of $x(t)$.

**Approach for an aperiodic signal with infinite support** If the signal has infinite support, divide the support into segments $[0, T]$, $[T, 2T]$, $[2T, 3T]$, apply the above approach for finite support to each segment. We obtain a sequence of spectra, one for each segment. This shows the spectra varies with time. There are lots of issues here—for example, what choice of $T$?