1 Goals

1. What is the spectrum of a signal?
   (a) Definition and representations
   (b) Motivation for studying spectra
2. How does one assess the spectrum of a given signal?

Outline of Coverage

(a) Rough definition of spectrum

(b) Motivation for studying spectra

(c) Assessment of spectra of signals

- (i) a signal that is a sum of sinusoids
- (ii) a periodic signal via Fourier series
- (iii) segments of a signal

Materials

1. text: Sections 3.1–3.4 and 3.7
2. Authors' handout: 3.4.5
3. Prof. Wakefield's handout F2001: Sections 1 and 2

You are alerted to the corrections to Chapter 3 in errata!

2 What is the spectrum?

Situation When a signal is viewed as a sum of "elementary" signals, we can think about how much of all these elementary signals is contained in the signal.

Rough Definition I Roughly speaking, the "spectrum of a signal" is a collection of elementary signals with their distribution, of which the signal consists.

(a) (Common elementary signals)

- (i) sinusoids
- (ii) complex exponentials

(b) (Sinusoidal components) each individual signal

(c) (Complex exponential components) each individual signal

3 Description of Spectra

$$
\begin{aligned}
x(t) &= \sum \text{ Sinusoids} \\
&= \sum \text{ Complex exponentials}
\end{aligned}
$$

Example 3.1 Sum of sinusoids

$$x(t) = 4 + 3\cos(2\pi 800t) - 5\cos(2\pi 1000t).$$

It is noted that $x(t)$ consists of

4 units	dc
$3\angle 0$ units	$\cos(2\pi 800t)$
$5\angle(-\pi)$ units	$\cos(2\pi 1000t)$

Example 3.2 Sum of complex exponentials

$$
\begin{aligned}
x(t) &= 4 + 3\cos(2\pi 800t) - 5\cos(2\pi 1000t) \\
&= 4 + 3\Big(\frac{e^{j2\pi 800t} + e^{-j2\pi 800}}{2}\Big) - 5\Big(\frac{e^{j2\pi 1000t} + e^{-j2\pi 1000t}}{2}\Big) \\
&= 4 + \frac{3}{2}e^{j2\pi 800t} + \frac{3}{2}e^{j2\pi 800t} - \frac{5}{2}e^{j2\pi 1000t} - \frac{5}{2}e^{j2\pi 1000t}.
\end{aligned}
$$

It is noted that $x(t)$ consists of

4 units	dc
$\frac{3}{2}$ units	$e^{j2\pi 800t}$
$\frac{3}{2}$ units	$e^{-j2\pi 800t}$
$-\frac{5}{2}$ units	$e^{j2\pi 1000t}$
$-\frac{5}{2}$ units	$e^{j2\pi 1000t}$

3.1 Spectrum as a function

Often the spectrum is described by a function of sinusoidal or complex exponential frequency.

Example 3.3 Sum of complex exponentials

$$x(t) = 4 + \frac{3}{2}e^{j2\pi 800t} + \frac{3}{2}e^{j2\pi 800t} - \frac{5}{2}e^{j2\pi 1000t} - \frac{5}{2}e^{j2\pi 1000t}.$$

Frequency in Hz	Function Value
0	4
800	$\frac{3}{2}$
-800	$\frac{3}{2}$
1000	$-\frac{5}{2}$
-1000	$-\frac{5}{2}$

3.2 Rough Definition II

(a) (Sinusoidal components)

(i) The "spectrum of a signal" indicates how the signal may be thought of as being composed of sinusoids.

(ii) It describes the frequencies, amplitudes and phases of the sinusoids that "sum" to yield the signal.

(iii) The individual sinusoids that sum to give the signal are called "sinusoidal components".

(iv) The spectrum describes the distributions of amplitude and phase vs. frequency of the sinusoidal components.

(b) (Complex exponential components) Sinusoids can be decomposed into the sum of two complex exponentials

(i) The spectrum equivalently indicates how the signal may be thought of as being composed of complex exponentials.

(ii) It describes the frequencies, amplitudes and phase of the complex exponentials that "sum" to yield the signal.

(iii) The individual complex exponentials that sum to give the signal are called "complex exponential components".

(iv) Alternatively, the spectrum describes distribution of amplitude and phase vs. frequency of the complex exponential components.

(c) Sinusoidal and complex exponential components are also called "spectral components".

3.3 Plotting Double-Sided Spectra

(a) Plot lines at the frequencies of the exponential components (at both positive and negative frequencies).

(i) The height of the line is the magnitude of the component.

(ii) We label the line with the complex amplitude of the component, e.g. with $\frac{5}{2}e^{j\pi}$ at $f = 1000$.

(b) Alternatively, sometimes we make two line plots, one showing the magnitudes of the components and the other showing the phases.

4 Spectra of Real World Signals

- Speech
- Musical instrument
- Image

5 Spectra of Sinusoids

Given $x(t)$, a sum of sinusoids,

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k),$$

its double sided spectrum is the list (collection) of pairs (C_k, f_k) for $k = 0, \pm 1, \pm 2, \ldots$, where

$$C_0 = A_0,$$

$$C_k = \begin{cases} \frac{A_k e^{j\phi_k}}{2}, & k > 0, \\ \frac{A_k e^{-j\phi_k}}{2}, & k > 0. \end{cases}$$

The spectrum, as a function of frequency, then is given by

$$X(f) = \begin{cases} C_k, & f = f_k, \\ 0, & \text{otherwise.} \end{cases}$$