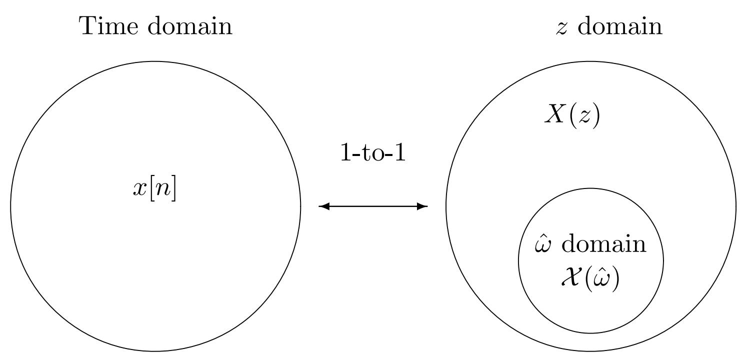
1 Benefits of the z-Transfom Approach

The frequency $\hat{\omega}$ domain approach can be generalized to the z domain.

(a) More signals can be dealt with.

(b) The transient and steady-state responses are described by one formula.

(c) Better analysis and design of filters are possible.



2 The z-Transform

2.1 Definition

- Definition: The (unilateral) z-transform of a discrete-time signal $w[n]$ is the function of z

$$W(z) = \sum_{n=0}^{\infty} w[n] z^{-n},$$

whose domain is the set of complex numbers ($z \in \mathbf{C}$) or a subset.

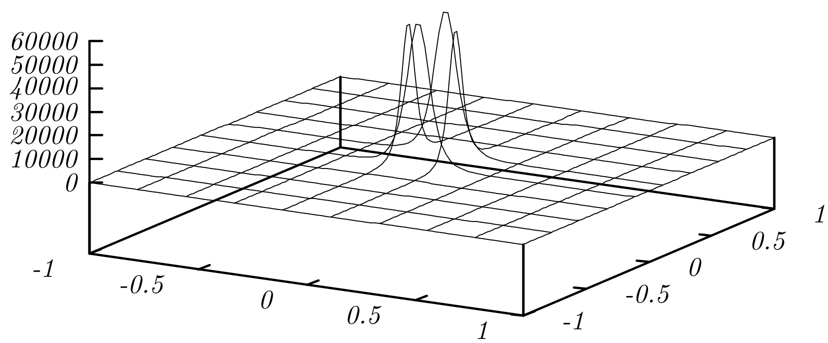
(a) $W(z)$ is, in general, a complex valued function.

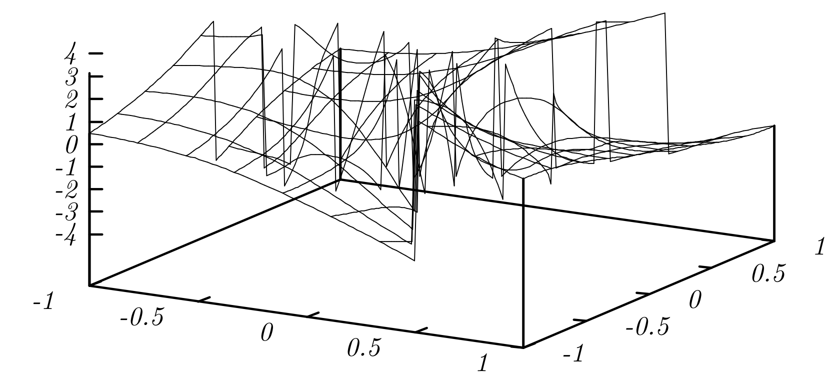
(b) The sequence $\ldots, w[-2], w[-1], w[0], w[1], w[2], \ldots$ *transforms* to a function $W(z)$. Here, $W(z)$ refers not simply to the function value at a specific z, but to the entire function, just like $w[n]$ refers here not to a specific term of the sequence but to the entire sequence.

- Example:

$$w[n] = \delta[n] + 3\delta[n-3] - \delta[n-5],$$
$$W(z) = 1 + 3z^{-3} - z^{-5}.$$

The plot of $|W(z)|$ and $\angle W(z)$ for $z = x + jy$





2.2 Notes, Facts and Comments

(1) Terminology: the z-transform

In addition to saying that the function $W(z)$ is the z-transform of w[n], we also say that the z-transform is the operation of computing $W(z)$. We often say that we obtain $W(z)$ by *taking the z-transform of* $w[n]$.

(2) We often write $W(z) = \mathcal{Z}\{w[n]\}$ and sometimes write $w[n] \iff W(z)$.

(3) one-to-oneness: Sequences that are distinct for $n \geq 0$ have distinct z-transforms. Thus we can determine a sequence from its z-transform, at least for $n \geq 0$.

Example: What sequence(s) has z-transforms $W(z) = 2 - 3z^{-1} + z^{-4}$?

Note that we cannot determine $w[n]$, $n < 0$ from $W(z)$.

(4) Notice that $W(z)$ is defined by an infinite sum.

Recall from mathematics that infinite sums are *not* always considered to be *defined* or to *exist* or to *converge*. (In this context, "defined", "exist" and "converge" are synonymous terms. People vary in their usage.) Thus, $W(z)$ is not always defined (or does not always exist, or does not always converge) for all z. More specifically,

(i) for some $w[n]$, $W(z)$ is defined for all z.
(ii) for some $w[n]$, $W(z)$ is not defined for any z's.
(iii) For some $w[n]$, $W(z)$ is defined for some z's but not others.

The region of the complex plane for which $W(z)$ is defined is called the **region of convergence** (ROC). Region of convergence issues add complexity to the discussion. (The region of convergence issues are not mentioned in the text.) We want you (the student) to be aware that there are such issues. But since this is just a first treatment, we will generally try to steer clear of such issues.

(5) If $w[n]$ has finite support, then $W(z)$ is well defined for every complex number z, except, possibly, $z = 0$, i.e. *the region of convergence is the entire complex plane except possibly the origin.*

(6) If $w[n]$ has support extending to $+\infty$, then for some z's the sum defining $W(z)$ might not converge, in which case $W(z)$ is not defined for all z. In particular, there will be some number $r > 0$ such that $W(z)$ is defined only for z such that $|z| > r$. That is, the ROC is the exterior of a circle with radius r.

(7) For a sequence that is zero after some finite time, $W(z)$ is a *polynomial* with terms that are negative powers of z, i.e. the sum of a finite number of powers of z^{-1} multiplied by coefficients. Thus, properties of polynomials become important to the study of filters. Recall the previous example:

$$\begin{aligned} w[n] &= \delta[n] + 3\delta[n-3] - \delta[n-5], \\ W(z) &= 1 + 3z^{-3} - z^{-5}. \end{aligned}$$

(8) For an infinite sequence starting at time $n = 0$ $W(z)$ is a *power series* in z^{-1} rather than a polynomial (infinitely many terms rather than finitely many terms).

$W(z)$ might be undefined for some values of z.

In the cases of interest to us, it will also be possible to write $W(z)$ as the ratio of two polynomials, as in

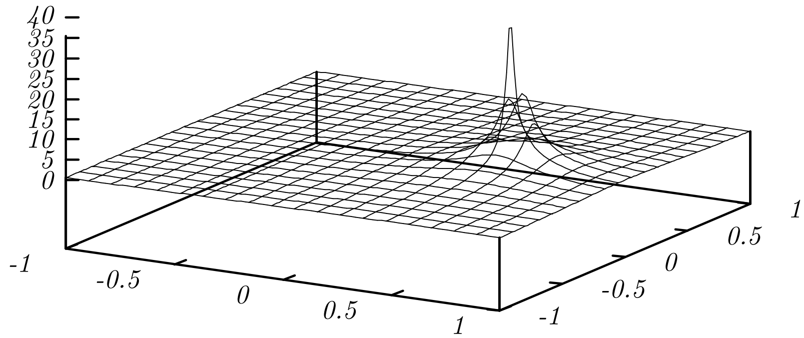
$$\begin{aligned} W(z) &= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots \\ &= \begin{cases} \frac{1}{1-z^{-1}} & |z| > 1, \\ \text{undefined} & |z| \le 1. \end{cases} \\ &= \frac{N(z)}{D(z)}. \end{aligned}$$

A function that is the ratio of two polynomials is called a **rational function**. (This kind of an extension of the concept of rational number, which is the ratio of two integers.)

Another example:

$$\begin{aligned} w[n] &= \delta[n-1] + \frac{1}{2}\delta[n-2] + \frac{1}{4}\delta[n-3] + \frac{1}{8}\delta[n-4] + \cdots . \\ W(z) &= z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{8}z^{-4} + \cdots \\ &= \begin{cases} \frac{z^{-1}}{1-\frac{1}{2}z^{-1}} & |z| > \frac{1}{2}, \\ \text{undefined} & |z| \le \frac{1}{2}. \end{cases} \end{aligned}$$

The plot of $|W(z)|$



(9) We will first apply the z-transform first to impulse response functions $h[n]$, and then to input and output signals. (Only later will we derive $Y(z) = X(z)H(z)$.)

(10) There is also something called the "bilateral z-transform"

$$W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n}.$$

In contrast, the transform we consider is sometimes called the *unilateral* z-transform.

(i) For a sequence $w[n]$ that is zero for $n < 0$, they give the same function of z. But for other sequences they do not.

(ii) We will consider only the unilateral z-transform.

(11) Since this is a first introductory discussion, we will not be giving a thorough discussion of z-transform, not even of unilateral z-transform. Rather we want to keep things simple. For example, we will try to avoid region of convergence issues.