1 The System Function of a Causal Filters

1.1 Definition

- Definition: The system function of a (causal) LTI filter is the z-transform of its impulse response $h[n]$.

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}.$$

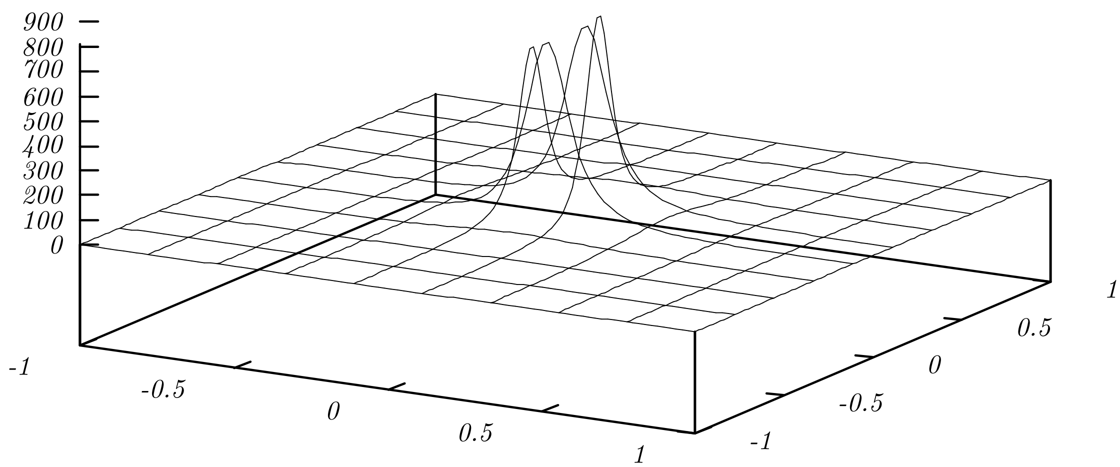
- It is a kind of generalized frequency response in that

$$\mathcal{H}(\hat{\omega}) = H(z)\,\Big|_{z=e^{j\hat{\omega}}} = H(e^{j\hat{\omega}}).$$

- Example: a causal FIR filter of order $M = 3$

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3],$$
$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3}.$$

The plot of $|H(z)|$



- Example: If a filter has system function $H(z) = z^{-3}$, what does it do?

  Its impulse response is $\delta[n-3]$. So it simply delays the input by 3 time units.

- Example: If a filter has system function $H(z) = 2 + 2z^{-1} + z^{-3}$, what does it do?

$$h[n] = 2\delta[n] + 2\delta[n-1] + \delta[n-3].$$

1.2 Notes

(1) The *system function* is often called the *transfer function*.

(2) We often write $H(z) = \mathcal{Z}\{h[n]\}$ and sometimes write

$$h[n] \iff H(z).$$

(3) $H(z)$ assigns a complex number to every complex number. That is, H(z) is a function with domain **C** and range **C**.

(4) It's hard to visualize a complex-valued function on the complex plane. So usually we visualize or plot its magnitude: $|H(z)|$. (We could also separately plot its phase.)

(5) For an **FIR filter** $H(z)$ is a **polynomial** in z^{-1}, i.e. the sum of a finite number of powers of z^{-1} multiplied by coefficients.

Since the sum is finite, it converges for all z (its well defined for all z), except possibly at $z = 0$ where terms like z^{-1} would be undefined.

(It converges at $z = 0$ iff $h[n] = 0$ all $n \geq 1$.)

(6) For an **IIR filter** H(z) is a **power series** and can also generally be written as the ratio of two polynomials—a **rational function** of z.

(Actually, for IIR filters as we define them, $H(z)$ is always the ratio of two polynomials.)

$H(z)$ might be undefined for some values of z other than 0, i.e. the sum might not converge for some values of z other than zero).

(7) Distinct causal impulse responses have distinct z-transforms. Therefore, $H(z)$ is a new characterization of filters.

As a reminder, previous characteristics include:

(i) FIR filter coefficients of a difference equation: b_k

(ii) The impulse response sequence: $h[n]$

(iii) The frequency response function: $\mathcal{H}(\hat{\omega})$

(iv) block diagram

We will need to become nearly as familiar with the system function as we are with the others. That is, when we think about a filter, we need to think about its system function.

(8) z-transforms can be applied to signals as well as to impulse responses, which we'll do later when we show

$$Y(z) = H(z)X(z) \quad \iff \quad y[n] = h[n] * x[n].$$

1.3 Important Facts

Fact 1:

$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}})$$

(a) This follows directly from definitions of $\mathcal{H}(\hat{\omega})$ and $H(e^{j\hat{\omega}})$.

$$\begin{aligned}\mathcal{H}(\hat{\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\hat{\omega}n} \\ &= \sum_{n=0}^{\infty} h[n]e^{-j\hat{\omega}n} \qquad \because \text{causality} \implies h[n] = 0 \text{ for } n < 0 \\ &= \sum_{n=0}^{\infty} h[n]z^{-n} \Big|_{z=e^{j\hat{\omega}}} = H(z) \Big|_{z=e^{j\hat{\omega}}} .\end{aligned}$$

(b) Thus the frequency response at frequency $\hat{\omega}$ equals the value of the system function at $z = e^{j\hat{\omega}}$, which is the *point in the z-plane on the unit circle at angle $\hat{\omega}$.*

(c) Thus, the system function $H(z)$ is a kind of extended frequency response function.

(d) Its values on the unit circle are the values of the frequency response function. But it assumes values outside the unit circle as well.

(e) This relationship is really important.

(f) It is emphasized that angle in the z-domain corresponds to frequency $\hat{\omega}$, because

$$z = e^{j\hat{\omega}} \qquad \iff |z| = 1, \quad \angle z = \hat{\omega}.$$

(g) The fact that frequency response can be found from the system function will be essential when dealing with IIR filters, because it will be the only way to find the frequency response function.

Fact 2: It is often useful to write the system function using only positive powers of z.

(The benefit of doing so comes in the next couple of facts.)

Example: $H(z) = 1 + z^{-1} + z^{-2} + z^{-3}$

By multiplying and dividing by z^3

$$H(z) = \frac{z^3 + z^2 + z + 1}{z^3}.$$

Fact 3: Using the fundamental theorem of algebra, for an FIR filter of order M, $H(z)$ can be written in the following form

$$H(z) = K\frac{(z - z_1) \cdots (z - z_M)}{z^M},$$

where $z_1, \ldots, z_M$ are the M-values (possibly complex) such that $H(z) = 0$ and K is real number.

They are called the **zeros** of $H(z)$ or of the filter and K is called the gain of $H(z)$ or of the filter. That is, we factor the numerator into a product of terms of the form $(z - z_i)$. Let's postpone discussion of the fundamental theorem of algebra and the general validity of Fact 3. Instead let's do an example and see how Fact 3 is used.

Example: for the filter in the previous example $H(z) = 1 + z^{-1} + z^{-2} + z^{-3}$.

$$H(z) = \frac{z^3 + z^2 + z + 1}{z^3}.$$

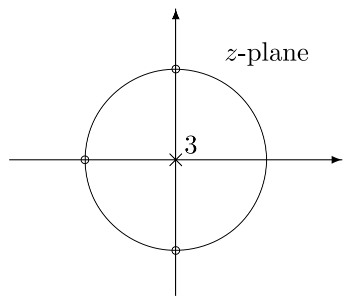
Find the roots of the numerator and factor the numerator:

$$z_1 = e^{j\pi/2} = j, \quad z_2 = e^{j\pi} = -1, \quad z_3 = e^{j3\pi/2} = -j.$$

Then

$$H(z) = \frac{(z-j)(z+1)(z+j)}{z^3},$$
$$K = 1.$$

Plot the zeros



Note: Matlab can find the roots of any polynomial, using the command `roots`.

Example: `roots([1 3 2])` for roots of $x^2 + 3x + 2 = 0$.

Impact of Zeros of the System Function

Now we wish to see how the zeros impact the frequency response, which, we recall, relates to the system function on the unit circle.

Fact 4: (Really important) For an FIR filter of order M,

$$|\mathcal{H}(\hat{\omega})| = |H(e^{j\hat{\omega}})| = K|e^{j\hat{\omega}} - z_1| \cdots |e^{j\hat{\omega}} - z_M|.$$

Fact 4 follows directly from Fact 3.

As will be illustrated soon, this demonstrates the relationship between the magnitude frequency response and the zeros.

Fact 5: (We won't use this much) For an FIR filter or order M,

$$\angle\mathcal{H}(\hat{\omega}) = \angle(e^{j\hat{\omega}} - z_1) + \cdots \angle(e^{j\hat{\omega}} - z_M) - M\hat{\omega}.$$

Fact 5 also follows directly from Fact 3.

Example: This shows what happens when zeros lie on the unit circle.

For the previous example,

$$H(z) = \frac{(z-j)(z+1)(z+j)}{z^3}.$$

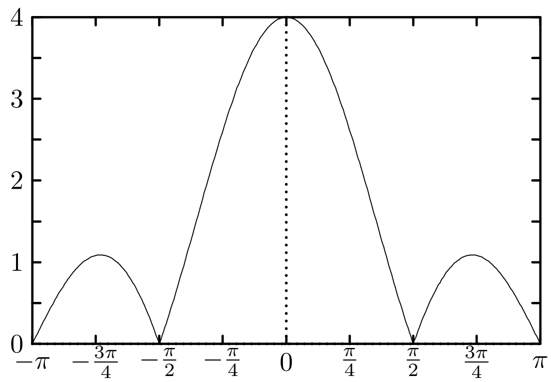
Therefore,

$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = \frac{(e^{j\hat{\omega}} - j)(e^{j\hat{\omega}} + 1)(e^{j\hat{\omega}} + j)}{e^{j3\hat{\omega}}}$$
$$= \frac{(e^{j\hat{\omega}} - e^{j\pi/2})(e^{j\hat{\omega}} - e^{j\pi})(e^{j\hat{\omega}} - e^{j3\pi/2})}{e^{j3\hat{\omega}}}$$

Thus,

$$|\mathcal{H}(\hat\omega)| = |H(e^{j\hat\omega})| = |e^{j\hat\omega} - e^{j\pi/2}| \cdot |e^{j\hat\omega} - e^{j\pi}| \cdot |e^{j\hat\omega} - e^{j3\pi/2}|.$$
$$\angle\mathcal{H}(\hat\omega)| = \angle H(e^{j\hat\omega}).$$

(i) Plot of $|\mathcal{H}(\hat\omega)|$.



(ii) Observe that $|\mathcal{H}(\hat\omega)|$ is zero at frequencies equal to the angles of the zeros.

(iii) Observe that $|\mathcal{H}(\hat\omega)|$ is nearly zero at frequencies nearly equal to the angles of the zeros.

(iv) Observe that $|\mathcal{H}(\hat\omega)|$ is large at frequencies away from these angles.

(v) Why is this? Look at the formula for $|\mathcal{H}(\hat\omega)|$. When $\hat\omega = \pi/2$, the first factor is zero. Similarly when $\hat\omega = \pi$ or $3\pi/2$.

(vi) More generally look at Fact 4.

$$\begin{aligned}|\mathcal{H}(\hat\omega)| &= K|e^{j\hat\omega} - z_1| \cdots |e^{j\hat\omega} - z_M| \\ &= K|e^{j\hat\omega} - r_1 e^{j\hat\omega_1}| \cdots |e^{j\hat\omega} - r_M e^{j\hat\omega_M}|.\end{aligned}$$

Observe that if $r_1 = 1$, as in our example, then when $\hat\omega = \hat\omega_1$, then $\mathcal{H}(\hat\omega) = 0$. And when $\hat\omega \approx \hat\omega_1$, then $\mathcal{H}(\hat\omega) \approx 0$.
The absence of a zero near $z = e^{j0} = 1$ implies that filter *passes* low frequencies.

In summary, Fact 4 leads to the following.

Fact 6: If a zero of the system function of a filter, say z_i, lies on the unit circle, i.e. $|z_i| = 1$, i.e. $z_i = e^{j\hat\omega_i}$, then the magnitude frequency response function has a null at frequency $\hat\omega = \hat\omega_i$. That is,

$$H(\angle z_i) = \mathcal{H}(\hat\omega_i) = 0.$$

Moreover the magnitude frequency response is small at nearby frequencies. That is,

$$\mathcal{H}(\hat\omega_i) \approx 0 \quad \text{for } \hat\omega \approx \hat\omega_i.$$

Moreover, if a zero, say z_i, lies near the unit circle, i.e. $|z_i| \approx 1$, i.e. $z_i = r_i e^{j\hat\omega_i}$, where $r_i \approx 1$, then the magnitude frequency response function is small at frequencies near $\hat\omega = \hat\omega_i$. That is,

$$\mathcal{H}(\hat\omega) \approx 0 \quad \text{for } \hat\omega \approx \angle z_i = \hat\omega_i.$$

Example Impact of zeros

(i) some zeros near the unit circle

(ii) some are on the unit circle

(iii) some are far from the unit circle

Discussion:

- Frequencies at which the frequency response function is zero are called **null frequencies** or simply **nulls**.
- Fact 6 shows that zeros of the system function on the unit circle cause nulls in the frequency response function at frequencies equal to the angles of the zeros.

  Moreover, zeros of the system function near the unit circle indicate the frequencies at which the system function will be small.

  Conversely, zeros that are not near the unit circle have little or no effect on the frequency response.
- The complete story, i.e. the complete magnitude frequency response of an FIR filter at frequency $\hat{\omega}$ is told by the products of terms

$$|\mathcal{H}(\hat{\omega})| = K|e^{j\hat{\omega}} - z_1| \cdots |e^{j\hat{\omega}} - z_M|.$$

- From this we see that $|\mathcal{H}(\hat{\omega})|$ is largest at frequencies $\hat{\omega}$ such that $e^{j\hat{\omega}}$ is far from all zeros. This tells us where the peaks are.

  Estimation of $|\mathcal{H}(\hat{\omega})|$ graphically is possible by measuring distances to zeros from $e^{j\hat{\omega}}$.
- This demonstrates two of the principal benefits of z-transforms for FIR filters.
  (i) Computing the z-transform of the impulse response of an FIR filter and then finding its zeros shows us precisely where its frequency response function is small.
  (ii) We can design an FIR filter by choosing zeros z_1, z_2, … on or near the unit circle at angles corresponding to frequencies at which we want the frequency response function to be small, then multiplying $(z - z_i)$ terms to find $H(z)$, then finding $h[n]$ from $H(z)$, then picking off the filter coefficients $\{b_k\}$. (Actually, one can find $\{b_k\}$ directly from $H(z)$.)
- At this point we see a real payoff to z-transforms!

1.4 Examples with System Functions of FIR Filters

1.4.1 Analysis of the running average filter of order M

- Find its system function $H(z)$ and its zeros.
- Predict the resulting null frequencies.
- Plot $|H(z)|$ and/or $|\mathcal{H}(\hat{\omega})|$.
- Compare the actual nulls the predicted ones.

1.4.2 Design of a notch FIR filter

Place one or more zero's at a desired frequency.

1.4.3 Design of a bandpass FIR filter

Place zeros on the unit circle in such a way as to leave a gap at some frequency.

1.5 Discussion of Fact 3

Now we go back and have some further discussion of the fundamental theorem of algebra and Fact 3. Specifically, Fact 3 comes from:

Fundamental Theorem of Algebra

Suppose we have a polynomial

$$P(z) = P_0 + P_1 z + \cdots + P_M z^M$$

of *degree* M, with P_i's real, but z allowed to be complex. Then

(a) $P(z) = P_M(z - z_1)(z - z_2)\cdots(z - z_M)$, where z_1, z_2, ..., z_M are the M values of z such that $P(z) = 0$. The z_i's are called the **roots** of $P(z)$.

(b) Some of the roots may be complex.

(c) $P(z)$ has exactly M roots, z_1,..., z_M, i.e, values such that $P(z) = 0$.

(d) Some of the roots may be the identical. If a certain root appears q times, then we say there are q roots with that value, or that it is a root of **multiplicity** q.

(e) If z_i is complex root, then there is some z_k such that $z_k = z_i^*$.

This theorem shows that a polynomial is *completely characterized* by

(i) P_M and

(ii) the places where it is zero, and the multiplicities of the zeros.

That is, if you fix the zero points and P_M, there is one and only one function that passes through them. It's kind of like a reconstruction from samples.

Poles: Notice that for an FIR filter, the denominator of the system function $H(z)$ also has zeros, namely, at $z = 0$. These are called the *poles* of the system function or filter. Poles will be important for IIR filters. But for FIR filters they are not important. Specifically, they have no effect on the filter magnitude or phase, because they are at the origin and are equidistant from all points on the unit circle.

Example: Notice that in our earlier example, the complex zeros came in conjugate pairs: $z_1 = j$, $z_2 = -1$, $z_3 = -j$.

Example: Redesign a notch filter so that $h[n]$ is real. Note that the previous design was not a "real" filter.

Example: Redesign a bandpass FIR filter pick our zeros and gain.

2 Summary of z-domain analysis of FIR filters

2.1 Analysis based on $H(z)$

Given an FIR filter, find the system function $H(z)$ and then find its zeros. This shows us exactly where the magnitude frequency response has nulls, and approximately, where it is small.

2.2 Design

We know that we can describe an FIR filter simply by specifying its gain and its zeros.

Thus if we want the magnitude frequency response to be zero or small at certain frequencies, we can design a filter that does this simply by placing zeros in appropriate places.

Specifically if we want nulls at frequencies $\hat{\omega}_1, \ldots, \hat{\omega}_M$,

(i) We make zeros $z_i = e^{j\hat{\omega}_i}$, $i = 1, \ldots, M$.

(ii) We make system function

$$H(z) = K\frac{(z - z_1)\cdots(z - z_M)}{z^M},$$

where we choose K any way we like.

(iii) To find impulse response or difference equation, we multiply terms to write $H(z)$ as polynomial in powers of z^{-1}.

(iv) We find impulse response $h[n]$ by picking off coefficients of $H(z)$.

(v) We find difference equation directly from $h[n]$.

This will make the frequency response small in the neighborhood of these frequencies as well.

If we don't want a null, but only want a *dip* in the frequency response at some specified frequency $\hat{\omega}_i$, we don't put zero right on the unit circle. Instead we choose

$$z_i = r_i e^{j\hat{\omega}_i},$$

where $r_i \neq 1$, but $r_i \approx 1$.

2.3 Notes

(a) **Characterizing FIR filters**

- (i) difference equation coefficients
- (ii) the impulse response
- (iii) block diagram
- (iv) the frequency response
- (v) the system function
- (vi) gain, poles, and zeros

(b) Same number of poles as zeros

(c) Poles at the origin are not significant because they effect all frequencies equally.

(d) Complex zeros come in conjugate pairs