1 More About z-Transform

1.1 Inverse z-Transform

- Definition:

  From the one-to-one property, we know that

  > for any z-transform $W(z)$, there is one and only one sequence $w[n]$ that has this z-transform and is zero for $n < 0$.

  This sequence is called the **inverse z-transform** of W(z).

  Also, the term inverse z-transform also refers to the process of finding $w[n]$.

- Approaches:

  There are three methods to find the inverse z-transform.

  (1) The "by inspection" method:
  This suffices when $W(z)$ is a polynomial with nonpositive powers of z. We've used this before.
  Example: The inverse z-transform of $W(z) = 1 + 3z^{-1} - z^{-5}$ is

  $$w[n] = \delta[n] + 3\delta[n-1] - \delta[n-5].$$

  (2) The "division" method:
  This works when $W(z)$ is a rational function, i.e. the ratio of two polynomials. By polynomial division, express $W(z)$ as power series with nonpositive powers of z. $w[n]$ is the coefficient of z^{-n}.
  Example:

  $$\begin{aligned} W(z) &= \frac{N(z)}{D(z)} \text{ or } \frac{N(z^{-1})}{D(z^{-1})} \\ &= \frac{1}{1-z^{-1}} \\ &= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots . \end{aligned}$$

  We get $w[n] = u[n]$.

  (3) The partial expansion method:
  To be discussed later. This also works when $W(z)$ is rational function. The advantage of the partial fraction method is that it leads to a closed form expression for $w[n]$, whereas division often gives us only the terms of $w[n]$ as far as we are willing to carry out the division.

  (4) The contour integration method:

1.2 Properties of z-Transform

(a) Linearity/superposition property

If $v[n] \iff V(z)$ and $w[n] \iff W(z)$, then

$$av[n] + bw[n] \iff aV(z) + bW(z).$$

(b) Time-delay property

If $w[n] \iff W(z)$ and $w[n] = 0$ for $n < 0$, then

$$w[n - n_0] \iff W(z)z^{-n_0}.$$

Derivation:

$$\begin{aligned}
\mathcal{Z}\{w[n-n_0]\} &= \sum_{n=0}^{\infty} w[n-n_0]z^{-n} \\
&= \sum_{m=-n_0}^{\infty} w[m]z^{-m-n_0} \quad \text{change of variables } m = n - n_0 \\
&= \Big(\sum_{m=0}^{\infty} w[m]z^{-m}\Big)z^{-n_0} \quad \because w[m] = 0, m < 0 \\
&= W(z)z^{-n_0}.
\end{aligned}$$

Example: $w[n] = 2\delta[n] + \delta[n-1] - 3\delta[n-3] \iff W(z) = 2 + z^{-1} - 3z^{-3}$.

Find the z-transform of $w[n-2] = 2\delta[n-2] + \delta[n-3] - 3\delta[n-5]$.

$$W(z)z^{-2} = 2z^{-2} + z^{-3} - 3z^{-5}.$$

Example: Suppose we have a filter with system function $H(z)$. What does the filter with system function $H(z)z^{-3}$ do?

Its impulse response is $h[n-3]$. So it filters with $h[n]$ and delays by 3.

(c) Some important z-transform pairs

People who work frequently with z-transforms typically keep a table at hand that records the z-transforms of commonly occurring time sequences. For example, the z-transforms of a couple of commonly sequences are listed below.

$w[n]$	$W(z)$	ROC
a^n	$\frac{1}{1-az^{-1}} = \frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$u[n]$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert > 1$
$e^{j\hat{\omega}n}$	$\frac{1}{1-e^{j\hat{\omega}}z^{-1}} = \frac{z}{z-e^{j\hat{\omega}}}$	$\lvert z\rvert > 1$
$\cos(\hat{\omega}n)$	$\frac{1-\cos(\hat{\omega})z^{-1}}{1-2\cos(\hat{\omega})z^{-1}+z^{-2}} = \frac{z^2-\cos(\hat{\omega})z}{z^2-2\cos(\hat{\omega})z+1}$	
$\cos(\hat{\omega}n+\phi)$	$\frac{\cos\phi-\cos(\hat{\omega}-\phi)z^{-1}}{1-2\cos(\hat{\omega})z^{-1}+z^{-2}} = \frac{\cos(\phi)z^2-\cos(\hat{\omega}-\phi)z}{z^2-2\cos(\hat{\omega})z+1}$	

(d) Convolution to multiplication property

Suppose $v[n]$ and $w[n]$ are sequences that are zero for $n < 0$, with z-transforms $V(z)$ and $W(z)$. Then

$$\mathcal{Z}\{v[n] * w[n]\} = V(z)W(z) \quad \text{or } v[n] * w[n] \iff V(z)W(z).$$

(i) This is another very useful property of the z-transforms.
(ii) We'll apply it and then derive it.
(iii) This property need not be valid if $v[n]$ or $w[n]$ is not zero for some $n < 0$.

Application to Filters:

(i) Suppose $x[n]$ is the input to a (causal) filter with impulse response impulse response $h[n]$ (FIR or IIR) and system function $H(z)$.
(ii) Suppose $x[n] = 0$, $n < 0$.
(iii) Suppose $X(z)$ is the z-transform of $x[n]$.

Then the z-transform of the output $y[n]$ is

$$Y(z) = X(z)H(z),$$

for all z for which $X(z)$ and $H(z)$ are *both* defined.

Notes: This is, in effect, a new description of the input/output relationship of a filter. It is useful only for suddenly applied signals. For FIR filters it can replace the "hybrid" method described in the frequency response lectures and Chapter 6. But it will also work for IIR filters.

New z-transform method for determining the output of filter

Given $x[n]$ ($x[n] = 0$, $n < 0$) and $h[n]$ (causal).
Step 1: Find $X(z)$
Step 2: Find $H(z)$
Step 3: $Y(z) = X(z)H(z)$
Step 4: $y[n] = \mathcal{Z}^{-1}\{Y(z)\}$

Note: This is a "symbolic" or "by hand" approach. It's not a numerical approach, as is convolution.

Derivation of Convolution Property

(i) Derivation 1:

$$\begin{aligned}
y[n] = x[n] * h[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=0}^{\infty} h[k]x[n-k] \quad \because h[k] = 0, k < 0. \\
Y(z) = \mathcal{Z}\{y[n]\} &= \mathcal{Z}\Big\{\sum_{k=0}^{\infty} h[k]x[n-k]\Big\} \\
&= \sum_{k=0}^{\infty} h[k]\mathcal{Z}\big\{x[n-k]\big\} \because \text{linearity} \\
&= \sum_{k=0}^{\infty} h[k]X(z)z^{-k} \because \text{delay property} \\
&= X(z)\sum_{k=0}^{\infty} h[k]z^{-k} = X(z)H(z).
\end{aligned}$$

(ii) Derivation 2: Begin with the convolution description of the input/output relationship:

$$\begin{aligned} y[n] = x[n] * h[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=0}^{\infty} h[k]x[n-k] \quad \because h[k] = 0, k < 0. \end{aligned}$$

Take z-transform of y[n]

$$\begin{aligned} Y(z) = \mathcal{Z}\{y[n]\} &= \sum_{n=0}^{\infty} y[n]z^{-n} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{\infty} h[k]x[n-k] \Big) z^{-n} \\ &= \sum_{k=0}^{\infty} h[k] \sum_{n=0}^{\infty} x[n-k]z^{-n} \\ &= \sum_{k=0}^{\infty} h[k] \sum_{m=-k}^{\infty} x[m]z^{-m-k} \quad \because \text{change of variables } m = n-k \\ &= \sum_{k=0}^{\infty} h[k] \sum_{m=0}^{\infty} x[m]z^{-m-k} \quad \because x[m] = 0, m < 0 \\ &= \sum_{k=0}^{\infty} h[k]z^{-k} \sum_{m=0}^{\infty} x[m]z^{-m} \\ &= H(z)X(z). \end{aligned}$$

Recap: It is useful to think of there being "three" versions of the z-transform I/O law (like the "three laws of Ohm")

$$\begin{aligned} Y(z) &= X(z)H(z) \\ H(z) &= Y(z)X(z) \\ X(z) &= Y(z)H(z) \end{aligned}$$

We've already applied the first law. We'll use the second and third laws later, especially the second.