D.2 Examples

Example 1: Let us find the spectrum of the periodic signal with period 4 and

$$x[0] = 1,\; x[1] = 1,\; x[2] = 0,\; x[3] = 0$$

Since the signal has period 4 we may choose N = 4. Then from the DFT analysis formula we have

$$X[0] = \frac{1}{4}\left(x[0] + x[1]\, e^{-j\frac{2\pi}{4}0\cdot 1} + x[2]\, e^{-j\frac{2\pi}{4}0\cdot 2} + x[3]\, e^{-j\frac{2\pi}{4}0\cdot 3}\right)$$

$$= \frac{1}{4}(1 + 1 + 0 + 0) = \frac{1}{2}$$

$$X[1] = \frac{1}{4}\left(x[0] + x[1]\, e^{-j\frac{2\pi}{4}1\cdot 1} + x[2]\, e^{-j\frac{2\pi}{4}1\cdot 2} + x[3]\, e^{-j\frac{2\pi}{4}1\cdot 3}\right)$$

$$= \frac{1}{4}(1 + e^{-j\frac{\pi}{2}} + 0 + 0) = \frac{1}{4}(1-j) = \frac{\sqrt{2}}{4}\, e^{-j\pi/4}$$

$$X[2] = \frac{1}{4}\left(x[0] + x[1]\, e^{-j\frac{2\pi}{4}2\cdot 1} + x[2]\, e^{-j\frac{2\pi}{4}2\cdot 2} + x[3]\, e^{-j\frac{2\pi}{4}2\cdot 3}\right)$$

$$= \frac{1}{4}(1 + e^{-j\pi} + 0 + 0) = \frac{1}{4}(1-1) = 0$$

$$X[3] = \frac{1}{4}\left(x[0] + x[1]\, e^{-j\frac{2\pi}{4}3\cdot 1} + x[2]\, e^{-j\frac{2\pi}{4}3\cdot 2} + x[3]\, e^{-j\frac{2\pi}{4}3\cdot 3}\right)$$

$$= \frac{1}{4}(1 + e^{-j\frac{3\pi}{2}} + 0 + 0) = \frac{1}{4}(1+j) = \frac{\sqrt{2}}{4}\, e^{j\pi/4}$$

In summary,

$$X[0] = \frac{1}{2},\; X[1] = \frac{\sqrt{2}}{4}\, e^{j\pi/4},\; X[2] = 0,\; X[3] = \frac{\sqrt{2}}{4}\, e^{-j\pi/4}$$

and the spectrum is

$$\left\{ (\tfrac{1}{2}, 0),\; (\tfrac{\sqrt{2}}{4}\, e^{-j\pi/4}, \tfrac{\pi}{2}),\; (\tfrac{\sqrt{2}}{4}\, e^{j\pi/4}, \tfrac{3\pi}{2}) \right\}$$

The next four examples give the N-point DFT coefficients of periodic exponentials and sinsuosids. These can be found by inspection.

Example 2: An exponential with frequency that is a multiple of 1/N:

$$x[n] = e^{j(2\pi\frac{m}{N}n+\phi)} \quad \Leftrightarrow \quad X[k] = \begin{cases} e^{j\phi}, & k=m \\ 0, & k\neq m \end{cases}$$

X[k] can be computed by inspection, as in Section C.

Example 3: A cosine with frequency that is a multiple of 1/N:

$$x[n] = \cos(2\pi\frac{m}{N}\,n) \quad \Leftrightarrow \quad X[k] = \begin{cases} \frac{1}{2}, & k=m, N-m \\ 0, & \text{else} \end{cases}$$

X[k] can be computed by inspection after decomposing x[n] into complex exponentials via the inverse Euler formula, as in Section C.

Example 4: A sine with frequency that is a multiple of 1/N:

$$x[n] = \sin(2\pi\frac{m}{N}\,n) \quad \Leftrightarrow \quad X[k] = \begin{cases} \frac{1}{2j}, & k=m \\ -\frac{1}{2j}, & k=N-m \\ 0, & \text{else} \end{cases}$$

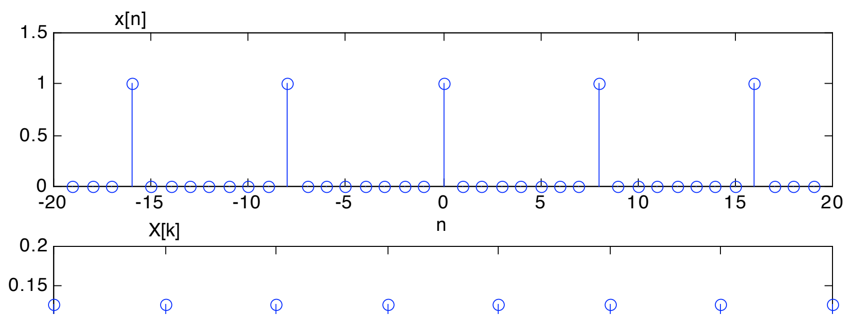
X[k] can be computed by inspection after decomposing x[n] into complex exponentials via the inverse Euler formula, as in Section C.

Example 5: A cosine with phase shift and frequency that is a multiple of 1/N

$$x[n] = \cos(2\pi\frac{m}{N}\,n+\phi) \quad \Leftrightarrow \quad X[k] = \begin{cases} \frac{1}{2}e^{j\phi}, & k=m \\ \frac{1}{2}e^{-j\phi}, & k=N-m \\ 0, & \text{else} \end{cases}$$

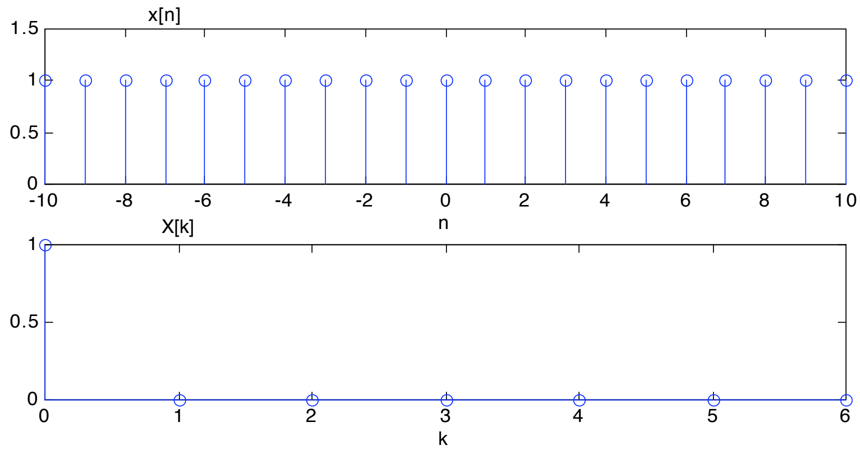
X[k] can be computed by inspection after decomposing x[n] into complex exponentials via the inverse Euler formula, as in Section C.

Example 6: $x[n] = \begin{cases} 1, & n = \text{multiple of } N \\ 0, & \text{else} \end{cases} \quad \Leftrightarrow \quad X[k] = \frac{1}{N},\ k = 0,\ldots,N-1$



(N=8)

Example 7: $x[n] = 1$, all $n \Rightarrow \quad X[k] = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$

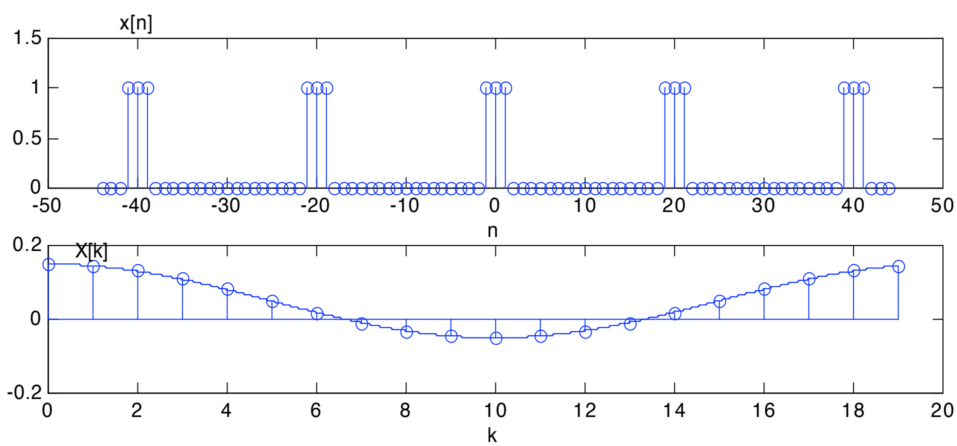


(N=7)

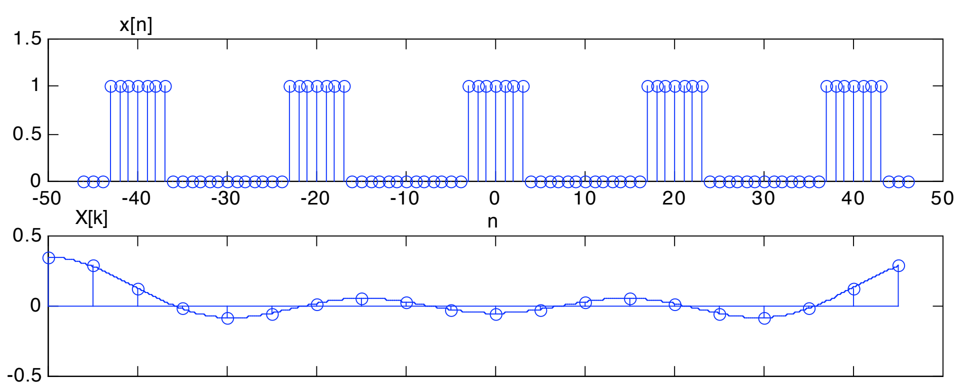
In this case, $X[k]$ is computed using the DFT analysis formula.

Example 8: $x[n]$ is periodic with fundamental period N and $x[n] = \begin{cases} 0, & -N/2 \le n \le -m-1 \\ 1, & -m \le n \le m \\ 0, & m+1 \le n \le N/2 \end{cases}$

$$\Rightarrow \quad X[k] = \frac{\sin((2m+1)\hat{\omega}/2)}{\sin(\hat{\omega}/2)} \bigg|_{\hat{\omega}=2\pi k/N}$$



(N=20,m=1)



(N=20,m=3)

Note that for these examples, $X[k]$ is real valued.

The smooth curve is the "envelope" of the DFT, which depends on N but not m.

It is interesting to notice that as the number of consecutive 1's increases, the spectrum becomes more concentrated at low frequencies. Conversely, as the number of consecutive 1's decreases, the spectrum becomes more spread out in frequency. This is representative of the "rule of the thumb" that generally speaking signals that are made of short pulses have more high frequency content than signals made of long pulses.

Example 9: Find the spectrum of the following signal:

$x[n]$ = sawtooth wave

Find a closed form expression for the coefficients.

Examine what happens to the spectrum as the parameters of the signal change.

Notice that for this signal, the spectrum cannot be computed by inspection. We very much need the analysis formula.

Example 10: Find the spectrum of the following example

$x[n]$ = finite sum of sinusoids.

In this example one computes the spectrum (i.e. the Fourier series coefficients) by inspection just as we did in Section C.2. The one-to-oneness of the relation between Fourier coefficients and periodic signals (see Fact D1 in Section D.3 below) means that the coefficients we obtain by inspection are the Fourier series coefficients.

Example 11: Find the signal corresponding to the following spectrum.

Show a spectrum with finite number of spectral lines. This is the same sort of problem as in the section on finite sums of sinusoids section.

Example 12: The signal shown below comes from someone speaking the vowel 'e'. It is nearly periodic. The magnitudes of its 128-point DFT and its magnitude spectrum are also shown.