Note: The $\Delta^2/12$ formula.

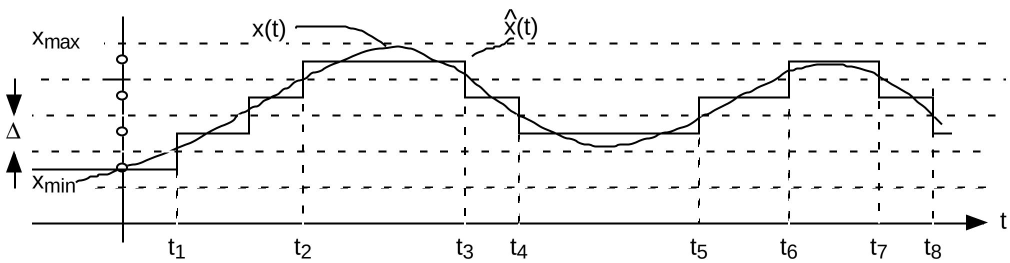
Fact: When an L-level uniform quantizer with range $[x_{min},x_{max}]$ and level spacing $\Delta = (x_{max}-x_{min})/L$ is used to quantize a data signal whose values lie mostly within the quantizer's range and whose standard deviation is much larger than Δ, the mean-squared error can be approximated as

$$\text{MSE} \cong \frac{1}{12}\Delta^2$$

Equivalently, the RMS error can be approximated as

$$\text{RMSE} \cong \sqrt{\frac{1}{12}\Delta^2}$$

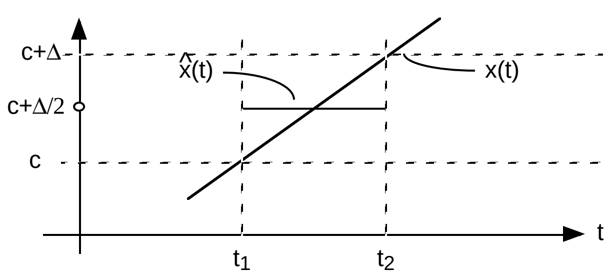
Derivation: Let us first show that $\Delta^2/12$ applies when quantizing (without sampling) a smoothly varying continuous-time signal $x(t)$ such as that illustrated below. Then we'll discuss how it also applies to other continuous-time signals, and to discrete-time signals as well. As an example, an $L=4$ level quantizer with level spacing $\Delta = (x_{max}-x_{min})/L$ is marked on the vertical axis, and the quantized version of $x(t)$ is also shown. The horizontal dashed lines mark the boundaries between the quantization segments and the vertical dashed lines mark where $x(t)$ passes through these boundaries, i.e. where the quantizer output changes from one level to another. The t_i's mark time intervals where the output $\hat{x}(t)$ is constant at a certain level.



Observation: When Δ is small, $x(t)$ is approximately a straightline in most of the time intervals where the output $\hat{x}(t)$ is constant .

We will now use this observation to show that $\text{MSE} \cong \Delta^2/12$ in such segments. Since $x(t)$ is nearly constant in most intervals it follows that $\text{MSE} \cong \Delta^2/12$ overall.

Accordingly, consider the idealized situation shown below, where a signal $x(t)$ is a straightline from time t_1, where $x(t_1) = c$ hits the lower boundary of a quantization segment, to time t_2, where $x(t_2) = c+\Delta$ hits the upper boundary of the segment, and where $x(t)$ is quantized to level $c+\Delta/2$ throughout this time interval.



The straightline has an equation: $x(t) = at + b$, where $x(t_1) = c$, $x(t_2) = c+\Delta$, and $a = \text{slope} = \Delta/(t_2-t_1)$. Let us now find the MSE in the time interval $[t_1,t_2]$:

$$\text{MSE} = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} \big(x(t) - (c+\Delta/2)\big)^2 dt = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} \big(at+b - (c+\Delta/2)\big)^2 dt$$

$$= \frac{1}{t_2-t_1} \int_{at_1+b-c-\Delta/2}^{at_2+b-c-\Delta/2} u^2 du, \qquad \text{where } u = at+b-c-\Delta/2,\ du = a\,dt$$

$$= \frac{1}{t_2-t_1} \int_{-\Delta/2}^{\Delta/2} u^2 du \, \frac{1}{a}, \qquad \text{because } at_1+b = x(t_1) = c,\ at_2+b = x(t_2) = c+\Delta$$

$$= \frac{1}{\Delta} \, \frac{1}{3} u^3 \Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12} \qquad \text{because } \frac{1}{a} = \frac{t_2-t_1}{\Delta}$$

Let's now ask what happens if $x(t)$ is not a smoothly varying signal like that shown above. With a little thought you can convince yourself that what really matters about $x(t)$ is not its shape, but its signal value distribution. In other words what matters is how frequently its various values occur. As discussed above, a smooth signal has the property that in most time intervals, it can be approximated by a straightline. And when it is a straightline over a time interval, the signal value distribution is constant over the corresponding range values, i.e. over the quantization segment! So the key idea is that the $\Delta^2/12$ formula applies whenever the signal value distribution is approximately constant over most quantization segments. (The constant value in one segment need not be the same as the constant value in another segment.)

The formula also applies to the quantization of discrete-time signals. Again the main idea is that the signal value distribution should be approximately constant over each quantization segment.