Part 3d: Spectra of Discrete-Time Signals

Outline

A. Definition of spectrum for discrete-time signals
B. Periodicity of discrete-time sinusoids
   and complex exponential signals
C. Spectra of signals that are *sums of sinusoids*
D. Spectra of *periodic signals*
  ◦ DFT (discrete Fourier transform)
  ◦ analysis / synthesis / properties
E. Spectra of *segments of signals* and *aperiodic signals*
F. Relationship between:
  ◦ spectrum of a continuous-time signal
  ◦ and spectrum of its samples
G. Bandwidth

Reading

- "Part 3d" lecture notes
- Text 4.1.1
- Do not read Chapter 9!
- Wakefield Fourier series & DFT "quick primer"

Introduction

Principal questions to be addressed:
- What, in a general sense, is the "spectrum" of a discrete-time signal?
- How does one assess the spectrum of a discrete-time signal?

Notes:
- As with continuous-time spectra, discrete-time spectra have two important roles:
  - Analysis and design. Spectra are theoretical tools that enable one to understand, analyze, and design signals and systems.
  - System component. The computation and manipulation of spectra is a component of many important systems.
- The motivation for studying the spectra of continuous-time signals was emphasized in the previous part of the course. A primary reason for our interest in the spectra of discrete-time signals is that when a discrete-time signal $x[n]$ is formed by sampling a continuous-time signal $x(t)$, the spectrum of $x[n]$ has a close relationship to the spectrum of the continuous-time signal $x(t)$.
- It is important to stress the similarity of the spectral concept for discrete-time signals to that for continuous-time signals.

Text Material

These lecture notes are intended to serve as text material for this section of the course. Though there is some discussion in Chapter 9 about the spectrum of discrete-time signals, it is not required or recommended reading. It does not give a general introduction to the concept of spectrum, and it introduces the DFT via a frequency-bank approach, which is very different than the Chapter 3 approach to Fourier series and to our approach to the DFT. Moreover, the DFT formulas in Chapter 9 differ by a scale factor from those that we use here and in the laboratory assignments.

These lectures introduce the concept for spectra of discrete-time signals with an "as similar as possible to continuous-time spectra" approach.

A. Rough definition of spectrum and motivation for studying spectra

A.1. Introduction to the concept of "spectrum"

This introduction parallels the introduction to spectrum for continuous-time signals.

Definition
Roughly speaking, the "spectrum" of a discrete-time signal is a representation of the signal as a sum of discrete-time complex exponentials.

(Note that for brevity we have jumped right to complex exponentials, rather than first indicating that we are interested in how signals are composed of sinusoids and subsequently splitting each sinusoid into two complex exponentials.)

- The spectrum describes the frequencies, amplitudes and phases of the discrete-time complex exponentials that combine to create the signal.
- The individual complex exponentials that sum to give the signal are called **complex exponential components**.
- The spectrum describes the distribution of amplitude and phase versus frequency of the complex exponential components.
- For real signals, pairs of exponentials sum to form sinusoids.
- Sinusoidal and complex exponential components are also called **spectral components**.

Plotting the spectra

We like to plot and visualize spectra. We plot lines at the frequencies of the exponential components. The height of the line is the magnitude of the component. We label the line with the complex amplitude of the component, *e.g.*, with $2e^{\jmath \pi/4}$.

Alternatively, sometimes we make two line plots, one showing the *magnitude* of each component and the other showing each *phase*. These are called the **magnitude spectrum** and **phase spectrum**, respectively.

Representations

Again we have three representations.

- Formula
- List of (frequency, complex amplitude) pairs
- Plot

A.2. Why are we interested in the spectra of discrete-time signals?

We are interested in the spectra of discrete-time signals for all the reasons that we are interested in the spectra of continuous-time signals. Presumably this does not require further discussion. However, the importance of spectra will be implicitly emphasized by the continued discussion and by continued examples of its application.

A.3. How does one assess the spectrum of a given signal?

As with continuous-time signals ...

- There is no single answer, *i.e.*, there is no universal spectral concept in wide use.
- The answer/answers do not fit into one course. We begin to address this question in EECS 206. The answer continues in EECS 306 and beyond.
- We use different methods to assess the spectrum of different types of signals.
  Specifically, in this section of the course, we will discuss
  - The spectrum of a sum of sinusoids (with support $(-\infty, \infty)$).
  - The spectrum of periodic signals (with support $(-\infty, \infty)$) via the **discrete-time Fourier series**, which will be called the **Discrete Fourier Transform** (**DFT**).
  - The spectrum of a segment of a signal via the DFT, which leads to the following.
    - The spectrum of aperiodic signals (not periodic) with finite support.
    - The spectrum of aperiodic signals with infinite support via the DFT applied to successive segments.
  - The relationship of the spectrum of a continuous-time signal to the spectrum of its samples.
- We won't discuss
  - The spectrum of a signal with infinite support and finite energy via the **discrete-time Fourier transform** (the **DTFT**, which is not the same as the DFT). This topic is discussed in EECS 451 and possibly in EECS 306.

B. Periodicity of discrete-time sinusoids and complex exponentials

Before discussing spectra of discrete-time signals in detail, we need to analyze the periodicity of discrete-time sinusoids and complex exponentials. There are a few wrinkles in discrete time that do not happen in continuous time.

B.1 Discrete-time sinusoids

The general discrete-time sinusoid is

$$x[n] = A\cos(\hat{\omega}n + \phi),$$

where n is an integer: $-\infty < n < \infty$.

- A is the **amplitude**.
  For standard form we use $A \geq 0$ and usually $A > 0$.
- ϕ is the **phase**.
  As with continuous-time signals, phase ϕ and phase $\phi + 2\pi$ are equivalent in the sense that

  $$A\cos(\hat{\omega}n + \phi) = A\cos(\hat{\omega}n + \pi + 2\pi), \qquad \forall n \in \mathbb{Z}.$$

  So for standard form we use $-\pi \leq \phi \leq \pi$.
- $\hat{\omega}$ is the **frequency**, sometimes called the **digital frequency**. The units of $\hat{\omega}$ are *radians per sample*.
  One could also write the sinusoid as $A\cos(2\pi\hat{f}n + \phi)$, where $\hat{f}$ is a frequency in *cycles per sample*.
  However, the radians-per-sample units are quite prevalent in digital signal processing, so we focus on that choice here.
  Each increment in time n increases $\hat{\omega}n$ by $\hat{\omega}$ radians.
  It is generally assumed that $\hat{\omega} \geq 0$ when describing discrete-time sinusoidal signals.

Key differences between discrete-time sinusoids and continuous-time sinusoids, forthcoming.
- In discrete-time, some sinusoids are not periodic!
- In discrete-time, there are "equivalent" frequencies.
- In discrete-time, $\hat{\omega}$ is not necessarily the fundamental frequency of the sinusoid!

Example.
As a prelude to subsequent analyses, let us attempt to examine the spectrum of the signal

$$x[n] = 2 + 3\cos\left(\frac{\pi}{4}n + \frac{\pi}{7}\right).$$

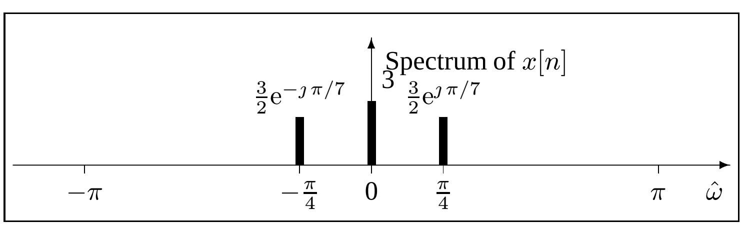
Following the continuous-time approach, we first decompose each sinusoidal signal into a sum of two complex exponential signals:

$$x[n] = 2 + \frac{3}{2}\mathrm{e}^{\jmath\,\pi/7}\mathrm{e}^{\jmath\,\frac{\pi}{4}n} + \frac{3}{2}\mathrm{e}^{-\jmath\,\pi/7}\mathrm{e}^{-\jmath\,\frac{\pi}{4}n}.$$

A natural definition of the spectrum is the following set of (complex amplitude, frequency) pairs:

$$\left\{\left(\frac{3}{2}\mathrm{e}^{-\jmath\,\pi/7}, -\frac{\pi}{4}\right),\ (2, 0),\ \left(\frac{3}{2}\mathrm{e}^{\jmath\,\pi/7}, \frac{\pi}{4}\right)\right\},$$

which we visualize as follows.



What about $x[n] = 2 + 3\cos(\frac{9\pi}{4}n + \frac{\pi}{7})$?

The preceding example is simple enough, but now consider plotting the spectrum of the continuous-time signal

$$x(t) = 2 + \cos\left(2\pi \frac{1}{5} t + \frac{\pi}{4}\right) + \cos\left(2\pi \frac{1}{5} t - \frac{\pi}{4}\right).$$

This signal has two terms with the same frequency so those two terms must be *combined* (using phasors) before plotting the spectrum: $x(t) = 2 + \sqrt{2}\cos\left(2\pi \frac{1}{5} t\right)$. Here it is "easy" to see which terms must be combined. ***(Picture)***
In discrete time, there can be many sinusoids with having equivalent frequencies that must be combined before plotting a spectrum!

Periodicity of discrete-time sinusoids

Fact B1
The discrete-time sinusoidal signal $A\cos(\hat{\omega} n + \phi)$ is **periodic** if and only if $\frac{\hat{\omega}}{2\pi}$ is a rational number, *i.e.*, if and only if we can write the **frequency** $\hat{\omega}$ in the form $\hat{\omega} = 2\pi M/N$ where M and N are integers.

If the rational number M/N is reduced so that the numerator and denominator have no common factors (except unity), then the **fundamental period** is the denominator N of the rational number.

In contrast, recall that for continuous-time signals, *every* sinusoid is periodic, and the **fundamental period** is simply the reciprocal of the frequency in Hz, and the frequency of a sinusoid is also its fundamental frequency.

Derivation
Recall the definition of periodicity:

A signal $x[n]$ is N-periodic if and only if

$$x[n+N] = x[n], \qquad \forall n \in \mathbb{Z}.$$

The **fundamental period** N_0 is the smallest such period.

Let us apply the definition to see when a discrete-time sinusoid is periodic. We want to know when there is an N such that

$$A\cos(\hat{\omega}(n+N) + \phi) = A\cos(\hat{\omega} n + \phi), \qquad \forall n \in \mathbb{Z}.$$

Since $A\cos(\hat{\omega}(n+N) + \phi) = A\cos(\hat{\omega} n + \hat{\omega} N + \phi)$, we see that the above equality holds when and only when $\hat{\omega} N = M \cdot 2\pi$, for some integer M, or equivalently, when and only when $\hat{\omega} = 2\pi \frac{M}{N}$. In other words, $\hat{\omega}$ must be 2π times a rational number.

Let us now find the fundamental period of $A\cos(\hat{\omega} n + \phi)$. If the sinusoid is periodic, then $\hat{\omega} = 2\pi \frac{K}{L}$ for some integers K and L. In this case, the sinusoid is periodic with period $N = L$ or $2L$ or $3L$ or ..., because for any such value of N, $\hat{\omega} N = 2\pi \frac{K}{L} N$ is an integer multiple of 2π.

What is the smallest period? If we eliminate any common factors of K and L, we can write $\hat{\omega} = 2\pi \frac{K'}{L'}$, where K' and L' have no common factors except unity (1). By the same argument as before, $A\cos(\hat{\omega} n + \phi)$ is periodic with period L'. This is the smallest possible period, so it is the fundamental period.

<u>Example</u>

(a) $A\cos(2\pi \frac{1}{2} n)$ is periodic with fundamental period $N_0 = 2$. The frequency of this signal is $\hat{\omega} = \pi$ radians per sample.

(b) $A\cos(2\pi \frac{3}{5} n)$ is periodic with fundamental period $N_0 = 5$. The frequency of this sinusoid is $\hat{\omega} = 2\pi \frac{3}{5} = \frac{6}{5}\pi$.

Notice that (b) has higher frequency than (a), yet (b) also has a longer fundamental period than (a).
This could not happen with continuous-time signals!

(c) $A\cos(2\pi \frac{4}{5} n)$ is periodic fundamental period $N_0 = 5$. The frequency of this sinusoid is $\hat{\omega} = 2\pi \frac{4}{5} = \frac{8}{5}\pi$.

Note that (b) and (c) have different frequencies, but the same fundamental period.
This could not happen with continuous-time signals!

(d) $A\cos(2n + \pi/2)$ is not periodic because $\hat{\omega} = 2\pi \frac{1}{\pi}$ is not 2π times a rational number.

(e) $A\cos(1.6\pi n)$ is periodic with fundamental period $N_0 = 5$, because $\hat{\omega} = 1.6\pi = 2\pi(0.8) = 2\pi \frac{4}{5}$.

Equivalent frequencies

Recall that phase ϕ and phase $\phi + 2\pi$ are "equivalent" in the sense that the following signals are equal:

$$A\cos(\hat{\omega}n + \phi) = A\cos(\hat{\omega}n + \phi + 2\pi), \qquad \forall n \in \mathbb{Z}.$$

As we now demonstrate, in the case of discrete-time sinusoids, there are also equivalent *frequencies*.

Fact B2
Frequency $\hat{\omega}$ and frequency $\hat{\omega} + 2\pi$ are "equivalent" in the sense that the following signals are equal:

$$A\cos((\hat{\omega} + 2\pi)n + \phi) = A\cos(\hat{\omega}n + 2\pi n + \phi) = A\cos(\hat{\omega}n + \phi), \qquad \forall n \in \mathbb{Z}.$$

This is another phenomena that is different for discrete time than for continuous time.
(The above derivation will not work if you try to replace $n \in \mathbb{Z}$ with $t \in \mathbb{R}$.)

Because of these equivalences, we usually limit attention to $0 \le \hat{\omega} < 2\pi$ for sinusoidal signals.

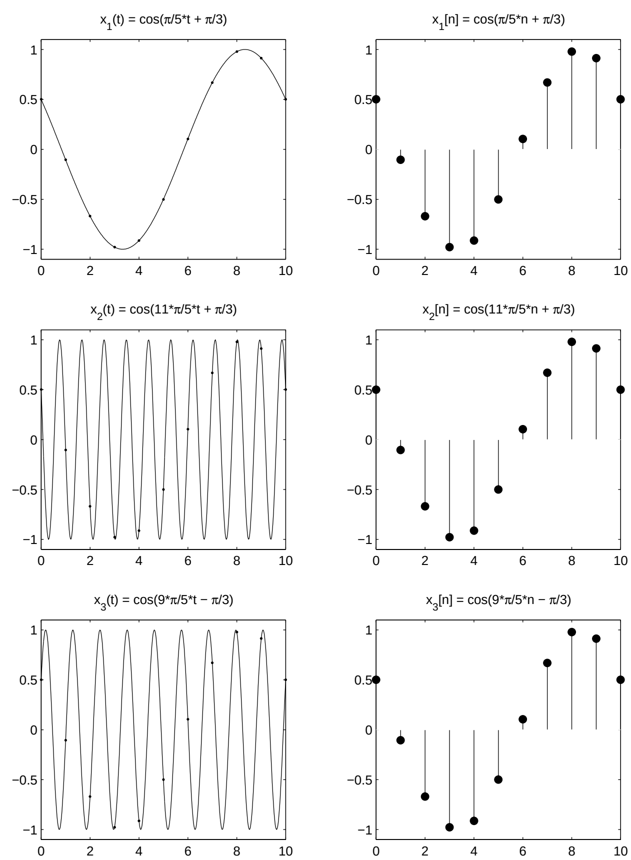
Example. The signals $\cos\left(\frac{1}{5}\pi n + \frac{\pi}{3}\right)$ and $\cos\left(\frac{11}{5}\pi n + \frac{\pi}{3}\right)$ are *identical* because $\frac{1}{5}\pi$ and $\frac{11}{5}\pi$ are **equivalent frequencies**.
Explanation:

$$\cos\left(\frac{11}{5}\pi n + \frac{\pi}{3}\right) = \cos\left(\left[\frac{1}{5}\pi + \frac{10}{5}\pi\right] n + \frac{\pi}{3}\right) = \cos\left(2\pi\frac{1}{5}n + 2\pi n + \frac{\pi}{3}\right) = \cos\left(2\pi\frac{1}{5}n + \frac{\pi}{3}\right).$$

Example. The signals $\cos\left(\frac{1}{5}\pi n + \frac{\pi}{3}\right)$ and $\cos\left(\frac{9}{5}\pi n - \frac{\pi}{3}\right)$ are *identical* because $-\frac{1}{5}\pi$ and $\frac{9}{5}\pi$ are **equivalent frequencies**.
Explanation:

$$\cos\left(\frac{9}{5}\pi n - \frac{\pi}{3}\right) = \cos\left(\frac{9}{5}\pi n - 2\pi n - \frac{\pi}{3}\right) = \cos\left(\left(\frac{9}{5}\pi - 2\pi\right) n - \frac{\pi}{3}\right) = \cos\left(-\frac{1}{5}\pi n - \frac{\pi}{3}\right) = \cos\left(\frac{1}{5}\pi n + \frac{\pi}{3}\right), \ \forall n \in \mathbb{Z}.$$

The following figure illustrates the difference between the continuous-time case and the discrete-time case.



B.2 Complex exponentials

The general discrete-time complex exponential is

$$A e^{\jmath \phi} e^{\jmath \hat{\omega} n}.$$

- A is the amplitude. We use $A \geq 0$.
- ϕ is the phase.
  Phase ϕ and phase $\phi + 2\pi$ are equivalent in the sense that

$$A e^{\jmath (\phi + 2\pi)} e^{\jmath \hat{\omega} n} = A e^{\jmath \phi} e^{\jmath 2\pi} e^{\jmath \hat{\omega} n} = A e^{\jmath \phi} e^{\jmath \hat{\omega} n}.$$

  So we use $-\pi \leq \phi \leq \pi$.
- $\hat{\omega}$ is the frequency. Its units are radians per sample. One could also write the exponential as $A e^{\jmath \phi} e^{\jmath 2\pi \hat{f} n}$, where $\hat{f}$ is frequency in cycles per sample.
  We allow $\hat{\omega}$ to be positive or negative. This is because we like to think of a cosine as being the sum of complex exponentials having positive and negative frequencies.

$$A \cos(\hat{\omega} n + \phi) = \frac{1}{2} A e^{\jmath \phi} e^{\jmath \hat{\omega} n} + \frac{1}{2} A e^{-\jmath \phi} e^{-\jmath \hat{\omega} n}.$$

Periodicity of discrete-time complex exponentials

Below, we list the periodicity properties of discrete-time exponentials. They are the same as discussed previously for discrete-time sinusoids.

Fact B3
$A e^{\jmath \phi} e^{\jmath \hat{\omega} n}$ is periodic when and only when $\hat{\omega}$ is 2π times a rational number.

If $\hat{\omega} = 2\pi M/N$ where M and N are integers having no common divisors, then the fundamental period is N.

Fact B4
Frequency $\hat{\omega}$ and frequency $\hat{\omega} + 2\pi$ are equivalent in the sense that

$$A e^{\jmath \phi} e^{\jmath (\hat{\omega} + 2\pi) n} = A e^{\jmath \phi} e^{\jmath \hat{\omega} n} e^{\jmath 2\pi} = A e^{\jmath \phi} e^{\jmath \hat{\omega} n}.$$

Discussion.

What do we make of the surprising fact that frequency $\hat{\omega}$ and frequency $\hat{\omega} + 2\pi$ are equivalent? We conclude that when we consider discrete-time sinusoids or complex exponentials, we can restrict frequencies to an interval of width 2π, since any other frequencies outside this interval will be redundant.

- Sometimes people restrict attention to $[-\pi, \pi)$ or $(-\pi, \pi]$ or perhaps $[-\pi, \pi]$.
- Sometimes people restrict attention to $[0, 2\pi)$.
- We'll do a bit of both.

C. The spectrum of a finite sum of discrete-time sinusoids

Our discussion of how to assess a spectrum parallels the discussion for continuous-time sinusoids. We begin by considering signals that are finite sums of sinusoids. However, because of the possibility of equivalent frequencies, there are a couple of key differences in how discrete-time and continuous-time spectra are assessed.

C.1. Example

A discrete-time signal that is a finite sum of sinusoids:

$$x[n] = 3 + 2\cos\left(\frac{1}{5}\pi n + \frac{\pi}{3}\right) + 5\cos\left(\frac{4}{3}\pi n\right) + 2\cos\left(\frac{9}{5}\pi n - \frac{\pi}{3}\right).$$

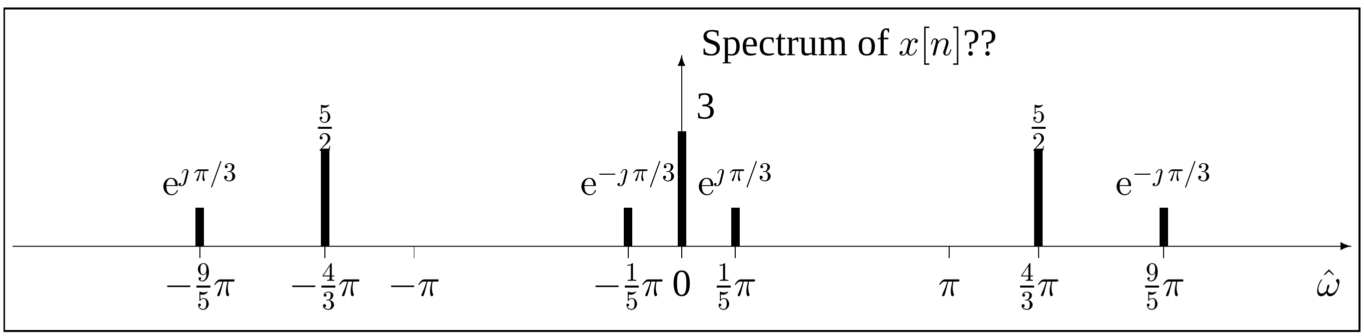
As in the continuous-time approach, we express $x[n]$ as a sum of complex exponentials:

$$\begin{aligned} x[n] &= 3 + \left(e^{\jmath\pi/3}e^{\jmath\frac{1}{5}\pi n} + e^{-\jmath\pi/3}e^{-\jmath\frac{1}{5}\pi n}\right) + \left(\frac{5}{2}e^{\jmath\frac{4}{3}\pi n} + \frac{5}{2}e^{-\jmath\frac{4}{3}\pi n}\right) \\ &\quad + \left(e^{-\jmath\pi/3}e^{\jmath\frac{9}{5}\pi n} + e^{\jmath\pi/3}e^{-\jmath\frac{9}{5}\pi n}\right). \end{aligned}$$

It would now seem natural to identify the spectrum as the following set of complex amplitude and frequency pairs:

$$\left\{\left(e^{\jmath\pi/3}, -\frac{9}{5}\pi\right), \left(\frac{5}{2}, -\frac{4}{3}\pi\right), \left(e^{-\jmath\pi/3}, -\frac{1}{5}\pi\right), (3,0), \left(e^{\jmath\pi/3}, \frac{1}{5}\pi\right), \left(\frac{5}{2}, \frac{4}{3}\pi\right), \ldots\right\},$$

and to draw the spectrum as follows.



However, some of these exponentials have equivalent frequencies, so the above plot is misleading! Specifically,

- Frequencies $-\frac{1}{5}\pi$ and $\frac{9}{5}\pi$ are equivalent because they differ by 2π (or a multiple of 2π).
- Frequencies $-\frac{9}{5}\pi$ and $\frac{1}{5}\pi$, are equivalent for the same reason.
- Frequency $\frac{4}{3}\pi$ is equivalent to $-\frac{2}{3}\pi$, and $-\frac{4}{3}\pi$ is equivalent to $\frac{2}{3}\pi$.

We combine all exponentials having equivalent frequencies (using phasor addition) into a single exponential component. In doing so, we get to choose which of the equivalent frequencies the resulting exponential component will have.

There are two possible conventions:

- Two-sided spectra. Exponential components have frequencies in the interval $[-\pi, \pi]$.
- One-sided spectra. Exponential components have frequencies in the interval $[0, 2\pi)$.

Choosing between these two conventions is mainly a matter of taste.

One-sided case

For the one-sided convention, we rewrite all of the complex exponentials in terms of equivalent frequencies that are in the interval $[0, 2\pi)$ and then combine phasors as follows:

$$\begin{aligned}
x[n] &= 3 + 2\cos\left(\frac{1}{5}\pi n + \frac{\pi}{3}\right) + 5\cos\left(\frac{4}{3}\pi n\right) + 2\cos\left(\frac{9}{5}\pi n - \frac{\pi}{3}\right) \\
&= 3 + \left(e^{\jmath\pi/3}e^{\jmath\frac{1}{5}\pi n} + e^{-\jmath\pi/3}e^{-\jmath\frac{1}{5}\pi n}\right) + \left(\frac{5}{2}e^{\jmath\frac{4}{3}\pi n} + \frac{5}{2}e^{-\jmath\frac{4}{3}\pi n}\right) \\
&\qquad\qquad + \left(e^{-\jmath\pi/3}e^{\jmath\frac{9}{5}\pi n} + e^{\jmath\pi/3}e^{-\jmath\frac{9}{5}\pi n}\right) \\
&= 3 + \left(e^{\jmath\pi/3}e^{\jmath\frac{1}{5}\pi n} + e^{-\jmath\pi/3}\underbrace{e^{\jmath\frac{9}{5}\pi n}}\right) + \left(\frac{5}{2}e^{\jmath\frac{4}{3}\pi n} + \frac{5}{2}\underbrace{e^{\jmath\frac{2}{3}\pi n}}\right) \\
&\qquad\qquad + \left(e^{-\jmath\pi/3}e^{\jmath\frac{9}{5}\pi n} + e^{\jmath\pi/3}\underbrace{e^{\jmath\frac{1}{5}\pi n}}\right) \\
&= 3 + \left(\underbrace{2e^{\jmath\pi/3}}e^{\jmath\frac{1}{5}\pi n} + \underbrace{2e^{-\jmath\pi/3}}e^{\jmath\frac{9}{5}\pi n}\right) + \left(\frac{5}{2}e^{\jmath\frac{4}{3}\pi n} + \frac{5}{2}e^{\jmath\frac{2}{3}\pi n}\right).
\end{aligned}$$

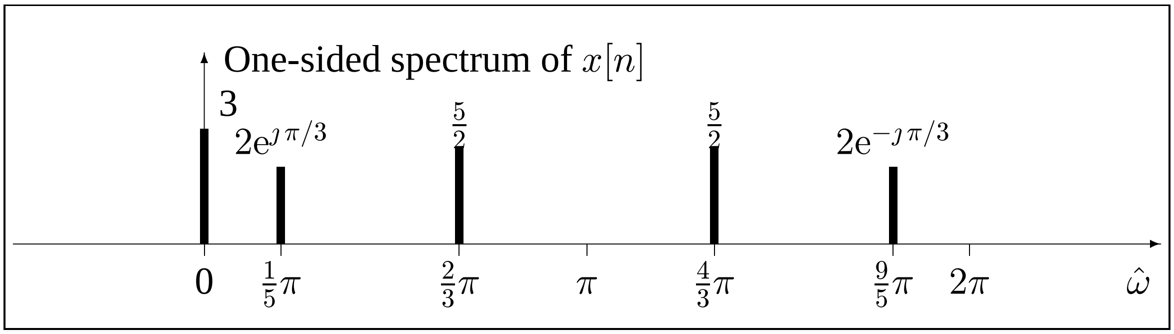
Repeating:

$$x[n] = 3 + \left(2e^{\jmath\pi/3}e^{\jmath\frac{1}{5}\pi n} + 2e^{-\jmath\pi/3}e^{\jmath\frac{9}{5}\pi n}\right) + \left(\frac{5}{2}e^{\jmath\frac{4}{3}\pi n} + \frac{5}{2}e^{\jmath\frac{2}{3}\pi n}\right)$$

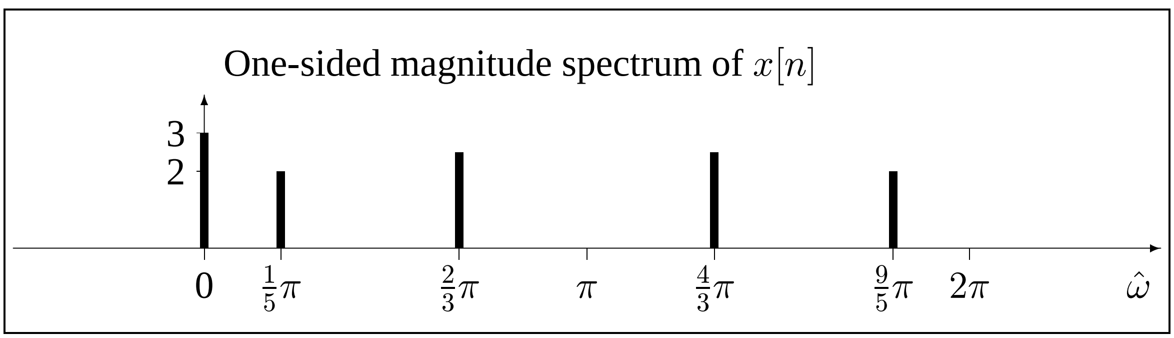
In terms of a list, we would write the one-sided spectrum of $x[n]$ as follows (*cf.* MP3):

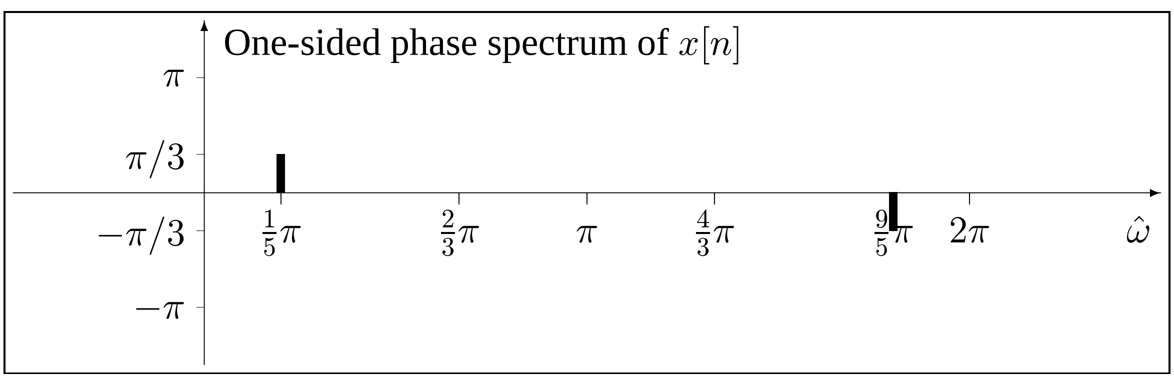
$$\left\{(3,0),\ \left(2e^{\jmath\pi/3}, \frac{1}{5}\pi\right),\ \left(\frac{5}{2}, \frac{2}{3}\pi\right),\ \left(\frac{5}{2}, \frac{4}{3}\pi\right),\ \left(2e^{-\jmath\pi/3}, \frac{9}{5}\pi\right)\right\}.$$

We draw the one-sided spectrum as follows.



Alternatively, the one-sided **magnitude spectrum** and **phase spectrum** are shown below.





Two-sided case

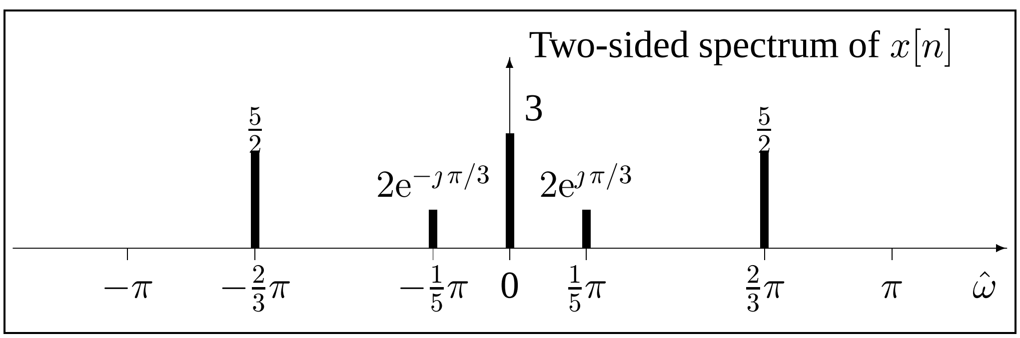
For the two-sided convention, we rewrite all of the complex exponentials in terms of the equivalent frequencies that are in the interval $[-\pi, \pi]$ and then combine phasors as follows:

$$\begin{aligned} x[n] &= 3 + \left(e^{\jmath\pi/3} e^{\jmath\frac{1}{5}\pi n} + e^{-\jmath\pi/3} e^{-\jmath\frac{1}{5}\pi n}\right) + \left(\frac{5}{2} e^{\jmath\frac{4}{3}\pi n} + \frac{5}{2} e^{-\jmath\frac{4}{3}\pi n}\right) \\ &\qquad + \left(e^{-\jmath\pi/3} e^{\jmath\frac{9}{5}\pi n} + e^{\jmath\pi/3} e^{-\jmath\frac{9}{5}\pi n}\right) \\ &= 3 + \left(e^{\jmath\pi/3} e^{\jmath\frac{1}{5}\pi n} + e^{-\jmath\pi/3} e^{-\jmath\frac{1}{5}\pi n}\right) + \left(\frac{5}{2} \underbrace{e^{-\jmath\frac{2}{3}\pi n}} + \frac{5}{2} \underbrace{e^{\jmath\frac{2}{3}\pi n}}\right) \\ &\qquad + \left(e^{-\jmath\pi/3} \underbrace{e^{-\jmath\frac{1}{5}\pi n}} + e^{\jmath\pi/3} \underbrace{e^{\jmath\frac{1}{5}\pi n}}\right) \\ &= 3 + \left(\underbrace{2e^{\jmath\pi/3}} e^{\jmath\frac{1}{5}\pi n} + \underbrace{2e^{-\jmath\pi/3}} e^{-\jmath\frac{1}{5}\pi n}\right) + \left(\frac{5}{2} e^{\jmath\frac{4}{3}\pi n} + \frac{5}{2} e^{-\jmath\frac{4}{3}\pi n}\right). \end{aligned}$$

In terms of a list, we express the **two-sided spectrum** as follows:

$$\left\{ \left(\frac{5}{2}, -\frac{2}{3}\pi\right), \left(2e^{-\jmath\pi/3}, -\frac{1}{5}\pi\right), (3, 0), \left(2e^{\jmath\pi/3}, \frac{1}{5}\pi\right), \left(\frac{5}{2}, \frac{2}{3}\pi\right) \right\}.$$

We draw the two-sided spectrum as follows.



Alternatively, we could draw the the magnitude spectrum and the phase spectrum.

Compare the one-sided and two-sided spectra above to see where the spectra lines go!

- One-sided spectra correspond naturally to the DFT.
- Two-sided spectra correspond naturally to the continuous-time case.

C.2. Spectrum of a general sum of discrete-time sinusoids

More generally, consider a signal of the form

$$\begin{aligned} x[n] &= A_0 + \sum_{k=1}^{N'} A_k \cos(\tilde{\omega}_k n + \phi_k) \\ &= A_0 + A_1 \cos(\tilde{\omega}_1 n + \phi_1) + \cdots + A_{N'} \cos(\tilde{\omega}_{N'} n + \phi_{N'}), \end{aligned}$$

where N', $A_0, A_1, \ldots, A_N$, $\tilde{\omega}_1, \ldots, \tilde{\omega}_N$, $\phi_1, \ldots, \phi_N$, are parameters that specify $x[n]$. We now rewrite this in several ways. First, using Euler's formula, we rewrite $x[n]$ as

$$x[n] = X_0 + \sum_{k=1}^{N'} \mathrm{Re}\{X_k e^{\jmath \tilde{\omega}_k n}\}$$

where the phasor corresponding to $A_k \cos(\tilde{\omega}_k n + \phi_k)$ is

$$X_k = A_k e^{\jmath \phi_k}.$$

(X_k is a complex number.)

Second, using the inverse Euler formula, we rewrite this as

$$x[n] = X_0 + \sum_{k=1}^{N'} \left[\frac{1}{2} X_k e^{\jmath \tilde{\omega}_k n} + \frac{1}{2} X_k^\star e^{-\jmath \tilde{\omega}_k n} \right].$$

To simplify this, we write it as follows:

$$x[n] = \sum_{k=-N'}^{N'} \beta_k e^{-\jmath \tilde{\omega}_k n}$$

where

$$\beta_0 = X_0 = A_0, \qquad \beta_k = \begin{cases} \frac{1}{2} X_k, & k = 1, \ldots, N' \\ \frac{1}{2} X_k^\star, & k = -1, \ldots, -N'. \end{cases}$$

Finally, as needed we combine terms with equivalent frequencies, to obtain

$$x[n] = \sum_{k=-N}^{N} \alpha_k e^{-\jmath \hat{\omega}_k n},$$

where $\{\hat{\omega}_k\}$ is a set of frequencies with values between $-\pi$ and π, and α_k is the phasor that is the sum of the appropriate β_k's corresponding to frequencies that are equivalent to $\hat{\omega}_k$:

$$\alpha_k = \sum_{j \,:\, \tilde{\omega}_j = \hat{\omega}_k + m2\pi, \; m \in \mathbb{Z}} \beta_j.$$

Note that

$$\alpha_{-k} = \alpha_k^\star, \qquad |\alpha_{-k}| = |\alpha_k|, \qquad \angle\alpha_{-k} = -\angle\alpha_k.$$

We now use this expression to make the following definition.

Definition. The spectrum of a sum of sinusoids.

The **two-sided spectrum** is

$$\begin{aligned} &\{(\alpha_{-N}, \hat{\omega}_{-N}), \ldots, (\alpha_{-1}, \hat{\omega}_{-1}), (\alpha_0, 0), (\alpha_1, \hat{\omega}_1), \ldots, (\alpha_N, \hat{\omega}_N)\} \\ &= \{(\alpha_N^\star, -\hat{\omega}_N), \ldots, (\alpha_1^\star, -\hat{\omega}_1), (\alpha_0, 0), (\alpha_1, \hat{\omega}_1), \ldots, (\alpha_N, \hat{\omega}_N)\}. \end{aligned}$$

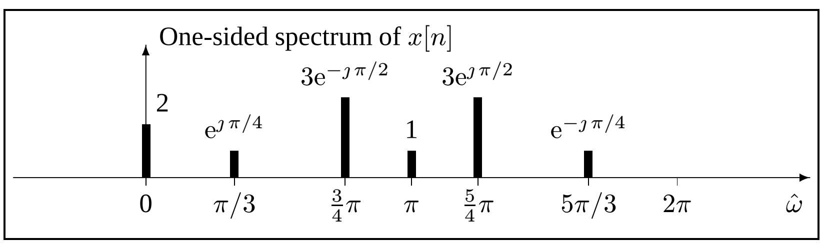
The **one-sided spectrum** is

$$\begin{aligned} &\{(\alpha_0, 0), (\alpha_1, \hat{\omega}_1), \ldots, (\alpha_N, \hat{\omega}_N) (\alpha_{-N}, \hat{\omega}_{-N} + 2\pi), \ldots, (\alpha_{-1}, \hat{\omega}_{-1} + 2\pi), \} \\ &= \{(\alpha_0, 0), (\alpha_1, \hat{\omega}_1), \ldots, (\alpha_N, \hat{\omega}_N) (\alpha_N^\star, 2\pi - \hat{\omega}_N), \ldots, (\alpha_1^\star, 2\pi - \hat{\omega}_1), \}. \end{aligned}$$

Notes. As with continuous-time spectra, we have the following properties.

- The spectrum, *i.e.*, one of these lists, is considered to be a simpler, more compact representation of the signal $x[n]$, *i.e.*, just a few numbers.
- The "spectrum" is often called the **frequency-domain representation** of the signal. In contrast, $x[n]$ is called the **time-domain representation** of the signal.
- The term $\alpha_k e^{\jmath \hat{\omega}_k n}$ is called the **complex exponential component** or **spectral component** of $x[n]$ at frequency $\hat{\omega}_k$.
- To obtain a useful visualization, we often plot the spectrum. That is, for each k, we draw a **spectral line** at frequency $\hat{\omega}_k$ with height equal to $|\alpha_k|$, and we label the line with the value of α_k, which is in general is complex.
- Alternatively, we sometimes separate the spectrum into magnitude and phase parts. For example, the two-sided versions of these are shown below.
  The **magnitude spectrum** is
  $$\begin{aligned}&\{(|\alpha_{-N}|, \hat{\omega}_{-N}),\ \ldots,\ (|\alpha_{-1}|, \hat{\omega}_{-1}),\ (|\alpha_0|, 0),\ (|\alpha_1|, \hat{\omega}_1),\ \ldots,\ (|\alpha_N|, \hat{\omega}_N)\}\\ &= \{(|\alpha_N|, -\hat{\omega}_N),\ \ldots,\ (|\alpha_1|, -\hat{\omega}_1),\ (|\alpha_0|, 0),\ (|\alpha_1|, \hat{\omega}_1),\ \ldots,\ (|\alpha_N|, \hat{\omega}_N)\},\end{aligned}$$
  which is even symmetric.
  The **phase spectrum** is
  $$\begin{aligned}&\{(\angle\alpha_{-N}, \hat{\omega}_{-N}),\ \ldots,\ (\angle\alpha_{-1}, \hat{\omega}_{-1}),\ (\angle\alpha_0, 0),\ (\angle\alpha_1, \hat{\omega}_1),\ \ldots,\ (\angle\alpha_N, \hat{\omega}_N)\}\\ &= \{(-\angle\alpha_N, -\hat{\omega}_N),\ \ldots,\ (-\angle\alpha_1, -\hat{\omega}_1),\ (\angle\alpha_0, 0),\ (\angle\alpha_1, \hat{\omega}_1),\ \ldots,\ (\angle\alpha_N, \hat{\omega}_N)\},\end{aligned}$$
  which is odd symmetric.
  And we might draw separate plots of the magnitude and phase. That is, for each k, the magnitude plot has a line of height $|\alpha_k|$ at frequency $\hat{\omega}_k$, and the phase plot has a line of height $\angle\alpha_k$ at frequency $\hat{\omega}_k$.
- Often, but certainly not always, we are more interested in the magnitude spectrum than the phase spectrum.

Example. Determine what signal $x[n]$ (in standard form) has the spectrum shown below.



Reading off the exponential components, we see that

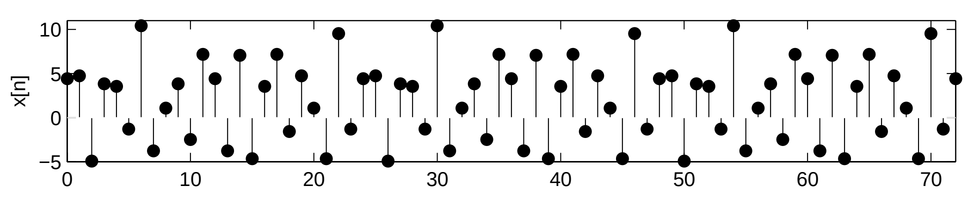
$$\begin{aligned}
x[n] &= 2 + e^{\jmath\pi/4} e^{\jmath\frac{\pi}{3}n} + 3e^{-\jmath\pi/2} e^{\jmath\frac{3\pi}{4}n} + 1e^{\jmath\pi n} + 3e^{\jmath\pi/2} e^{\jmath\frac{5\pi}{4}n} + e^{-\jmath\pi/4} e^{\jmath\frac{5\pi}{3}n}\\
&= 2 + \left(e^{\jmath\pi/4} e^{\jmath\frac{\pi}{3}n} + e^{-\jmath\pi/4} e^{\jmath\frac{5\pi}{3}n}\right) + \left(3e^{-\jmath\pi/2} e^{\jmath\frac{3\pi}{4}n} + 3e^{\jmath\pi/2} e^{\jmath\frac{5\pi}{4}n}\right) + e^{\jmath\pi n}\\
&= 2 + \left(e^{\jmath\pi/4} e^{\jmath\frac{\pi}{3}n} + e^{-\jmath\pi/4} e^{-\jmath\frac{\pi}{3}n}\right) + \left(3e^{-\jmath\pi/2} e^{\jmath\frac{3\pi}{4}n} + 3e^{\jmath\pi/2} e^{-\jmath\frac{3\pi}{4}n}\right) + e^{\jmath\pi n}\\
&= 2 + 2\cos(\frac{\pi}{3}n + \pi/4) + 6\cos(\frac{3\pi}{4}n - \pi/2) + \cos(\pi n).
\end{aligned}$$

Notice the use of equivalent frequencies in the middle of the derivation:

$$e^{\jmath\frac{5\pi}{3}n} = e^{-\jmath\frac{\pi}{3}n}.$$

Is $x[n]$ periodic? Yes, because all of the frequencies of the sinusoidal components are 2π times a rational number.

What is the period of $x[n]$? The three components have periods: 6, 8, and 2, the LCM of which is $N_0 = 24$.



The special case of $\hat{\omega} = \pi$.

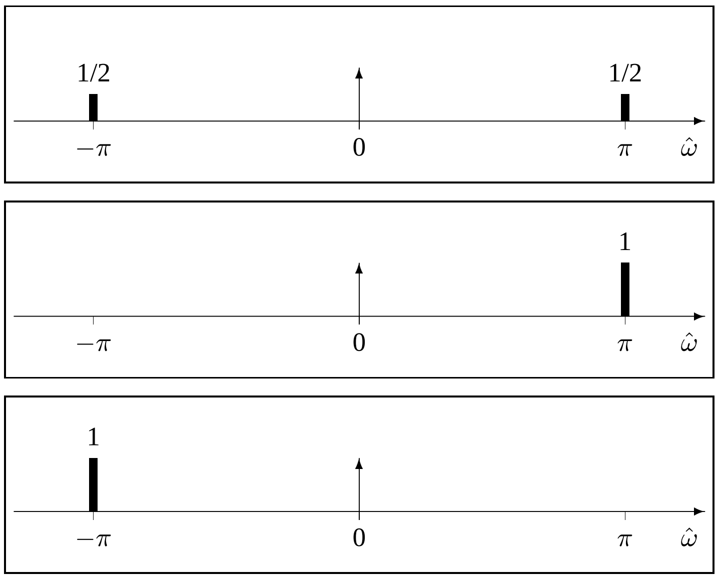
You may have noticed that we have mentioned that the two-side interval could be $[-\pi, \pi)$ or $(-\pi, \pi]$ or $[-\pi, \pi]$. The reason for this flexibility is that π and $-\pi$ are equivalent frequencies.

The following three figures show three equally valid two-sided spectra for the signal

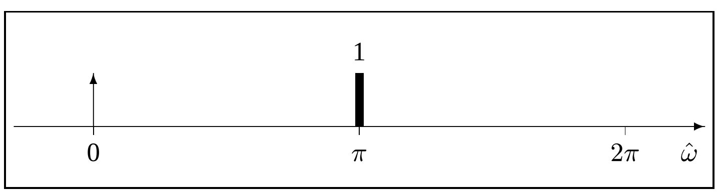
$$x[n] = \cos(\pi n) = (-1)^n.$$

The three figures correspond to the following three equivalent expressions for this particular signal:

$$x[n] = \frac{1}{2} e^{\jmath \pi n} + \frac{1}{2} e^{-\jmath \pi n} = e^{\jmath \pi n} = e^{-\jmath \pi n}.$$



In contrast, there is only one appropriate way to show the one-sided spectrum for this signal.



Exercise. Sketch the spectrum of the following signal

$$x(t) = \sqrt{2} \cos(\pi n + \pi/4).$$

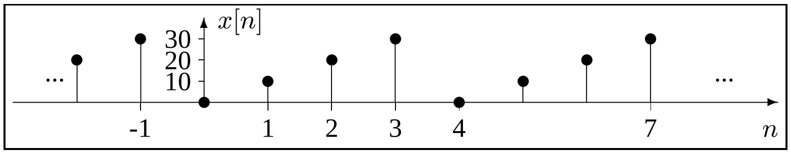
Surprised? Can you think of another signal that yields the same result?

This exercise should convince you that the frequency $\hat{\omega} = \pi$ requires special care.

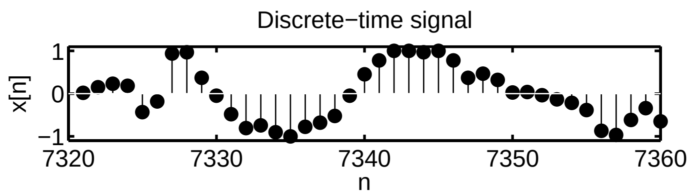
D. The spectrum of a periodic discrete-time signal

The preceding notes describe how to determine the spectra of signals that are sums-of-sinusoids.

But what is the spectrum of a (periodic) signal like the following?



Or, perhaps more interestingly, what is the spectrum of a sequence of samples of an analog signal, such as the following portion of an audio signal?



The answer to both of these questions is provided by the Discrete Fourier Transform (DFT).

Recall that from the Fourier series theorem we learned that the spectrum of a periodic continuous-time signal with period T is concentrated at frequencies that are multiples of $1/T$, and that the Fourier series analysis formula determines the specific component at each of these frequencies. In this section we learn of an analogous theorem that indicates that the spectrum of a periodic discrete-time signal with period N is concentrated at frequencies that are multiples of $2\pi/N$. The theorem also provides an analysis formula for determining the specific component at each of these frequencies.

D.1 The DFT Theorem

The spectrum of a discrete-time periodic signal derives from the following theorem.

The **Discrete Fourier Transform** (**DFT**) Theorem (aka The Discrete-Time Fourier Series Theorem)

Any periodic signal $x[n]$ with period N can be written as a sum of N complex exponentials with frequencies

$$0,\ \frac{2\pi}{N}1,\ \frac{2\pi}{N}2,\ \ldots,\ \frac{2\pi}{N}(N-1). \tag{3d-1}$$

Specifically, there are N **DFT coefficients**, denoted $X[0]$, $X[1]$, $\ldots$, $X[N-1]$ such that $x[n]$ can be expressed by the following **synthesis formula**:

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N} kn}, \qquad \forall n \in \mathbb{Z}.$$

The DFT coefficients are determined from the signal $x[n]$ via the following **analysis formula**:

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} kn}, \qquad k = 0, \ldots, N-1.$$

Note that frequencies $\frac{2\pi}{N}k$ and $\frac{2\pi}{N}(k+N)$ are equivalent frequencies, so we do not need terms with $k = N$ or $k = N+1$ etc.

Notes

- We will derive this theorem later. Unlike the Fourier Series Theorem, its derivation is well within the scope of this class!
- The term $e^{\jmath \frac{2\pi}{N} kn}$ appearing in the synthesis formula is the **complex exponential component** (equivalently, the **spectral component**) of $x[n]$ at frequency $\frac{2\pi}{N} k$.
- The theorem says that *any* periodic discrete-time signal can be represented as the sum of at most N complex exponential components with frequencies coming from the set $\left\{0, \frac{2\pi}{N}1, \frac{2\pi}{N}2, \ldots, \frac{2\pi}{N}(N-1)\right\}$.

  This means that the spectrum of a periodic signal with period N is concentrated at these frequencies (or a subset thereof).
- The synthesis formula is very much like the synthesis formula for the Fourier series of a continuous-time periodic signal, except that only a *finite* number of frequencies/exponential components are used. This stems from the fact that for discrete-time complex exponentials, every frequency outside the range $[0, 2\pi)$ is equivalent to some frequency within the range $[0, 2\pi)$.

  Furthermore, all of the information about a discrete-time periodic signal is contained in the N signal values over one period: $x[0], \ldots, x[N-1]$. Since N values is enough to describe $x[n]$, it is logical to expect that we should need at most N frequency components to describe $x[n]$. (If more than N frequency components were needed, then the spectrum of a signal would be a less concise description of the signal that the time-domain values, which would greatly diminish its utility!)
- For periodic signals, it is natural then to use the following list as the definition of the **one-sided spectrum**:

$$\left\{(X[0], 0), \left(X[1], \frac{2\pi}{N}1\right), \left(X[2], \frac{2\pi}{N}2\right), \ldots, \left(X[N-1], \frac{2\pi}{N}(N-1)\right).\right\}$$

  Thus, finding the spectrum of a periodic discrete-time signal involves finding its period and finding the $X[k]$'s.

  We could also define a **two-sided spectrum** from the DFT coefficients. However, as described later, the two-sided spectrum is somewhat messier.
- When we compute the $X[k]$'s using the analysis formula, there is no need to combine exponential components with equivalent frequencies, as we did previously when finding the spectrum of a finite sum of sinusoids. In essence, the DFT analysis formula has already done this for us.
- To aid the understanding of the synthesis and analysis formulas, it can be useful to view them in long form:

  The **synthesis formula**:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{\jmath \frac{2\pi}{N} kn} = X[0] + X[1] e^{\jmath \frac{2\pi}{N} 1n} + X[2] e^{\jmath \frac{2\pi}{N} 2n} + \cdots + X[N-1] e^{\jmath \frac{2\pi}{N}(N-1)n}.$$

  The **analysis formula**:

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-\jmath \frac{2\pi}{N} kn} = \frac{1}{N}\left(x[0] + x[1] e^{-\jmath \frac{2\pi}{N} 1n} + x[2] e^{-\jmath \frac{2\pi}{N} 2n} + \cdots + x[N-1] e^{-\jmath \frac{2\pi}{N}(N-1)n}\right).$$

  Notice how similar these formulas are; they only differ by the $1/N$ and by the sign in the exponent.

  An elementary example of computing these formulas is given in Section D.2.
- The frequency $2\pi/N$ is called the **fundamental** or **first harmonic** frequency.
  The frequency $\frac{2\pi}{N} k$ is called the **kth-harmonic** frequency.
  The component at frequency $2\pi/N$ is called the **fundamental** or **first harmonic** component.
  The component at frequency $\frac{2\pi}{N} k$ is called the **kth-harmonic** component.
- The analysis formula may be viewed as operating on a periodic signal $x[n]$ (actually, just on $x[0], \ldots, x[N-1]$) and producing N DFT coefficients $X[0], \ldots, X[N-1]$. This operation is considered to be a **transform** of the signal $x[n]$ into the set of coefficients $X[0], \ldots, X[N-1]$. This is why **transform** appears in the name **Discrete Fourier Transform**.

  Similarly, the synthesis formula may be viewed as operating on Fourier coefficients $X[0], \ldots, X[N-1]$ and producing a signal $x[n]$. This operation is considered to be an **inverse transform**.

- It is customary to write $X[k]$ as a shorthand for $\{X[0], \ldots, X[N-1]\}$, just as the notation "$x[n]$" usually refers to an entire signal. So there are two possible meanings for the notation "$X[k]$": it could mean the kth coefficient, or it could mean the entire set of N coefficients.

- The term **Discrete Fourier Transform** (**DFT**) commonly refers both to the *process* of applying the analysis formula as well as to the coefficients $X[k]$ that result from this process. For example, people often say "$X[k]$ is the DFT of $x[n]$."

- The process of applying the analysis formula to $x[n]$ is often called "finding/taking the DFT of $x[n]$" or, sometimes, "DFT'ing $x[n]$." Similarly, the process of synthesizing $x[n]$ from the DFT coefficients $X[k]$ is often called "finding/taking the inverse DFT of $X[k]$," or "inverse DFT'ing $X[k]$."

- It is traditional to use $X[k]$ to denote the DFT of $x[n]$, $Y[k]$ to denote the DFT of $y[n]$, $X_1[k]$ to denote the DFT of $x_1[n]$, and so on.

- When N is a power of 2, *i.e.*, $N = 2^m$ for some $m \in \mathbb{N}$, there is fast algorithm for computing the DFT (*i.e.*, the analysis formula), called the **fast Fourier transform** (**FFT**). Because the synthesis formula and the analysis formula are so very similar, a slight variation on the FFT algorithm, called the **inverse FFT** can also be used to compute the synthesis formula. These algorithms have enabled the widespread use of the DFT in the analysis, design, and implementation of signals and systems. The FFT is one of the most important tools of modern signal processing and is at the core of many of the ubiquitous digital signal processing devices around us.

- The FFT algorithm is available in MATLAB through the commands `fft` and `ifft`. If you use these commands, be aware of a factor of N difference between our definition of the DFT and MATLAB's convention.
  - To compute our definition of the DFT (analysis), use: `X = 1/N * fft(x)`
  - To compute our definition of the inverse DFT (synthesis), use: `x = N * ifft(X)`

- Since the summand in the analysis formula is periodic with period N, the limits of the summation could be replaced with any interval of length N.

- If a signal has period N, then it also has period $2N$ and period $3N$ and so on. Thus, when applying DFT analysis, we can choose which period N to use. Usually, but certainly not always, we choose N to be the **fundamental period**. When we want to specify explicitly the value of N used, we will say "the N-point DFT." As discussed in Section D.4, using $2N$ or $3N$ (etc.) instead of N does not change the spectrum.

- The DFT Theorem applies both to real signals and to complex signals.

- Observe that according to the analysis formula, coefficient $X[k]$ is computed by correlating $x[n]$ with the complex exponential $e^{-\jmath \frac{2\pi}{N} kn}$ and then dividing by N. This N is the energy of one period of the exponential:

$$\sum_{n=0}^{N-1} \left| e^{-\jmath \frac{2\pi}{N} kn} \right|^2 = \sum_{n=0}^{N-1} 1 = N.$$

  Suggested reading. The discussion of "signal components" at the end of Section IIIB of the "Introduction to Signals" by DLN. This section will help one to understand why the analysis formula has the form that it has.

  In the terminology of that discussion:
  - $X[k]\, e^{\jmath \frac{2\pi}{N} kn}$ is the component of $x[n]$ that is like $e^{\jmath \frac{2\pi}{N} kn}$,
  - $X[k]$ measures the similarity of $x[n]$ to the exponential.

- The reader is encouraged to review the discussion of Fourier series for continuous-time signals and observe the similarities with the DFT for discrete-time periodic signals. The principal differences are
  - t is replaced by n
  - T is replaced by N
  - The DFT synthesis formula has only N terms.
  - The DFT analysis formula uses a sum rather than an integral.

DFT Examples

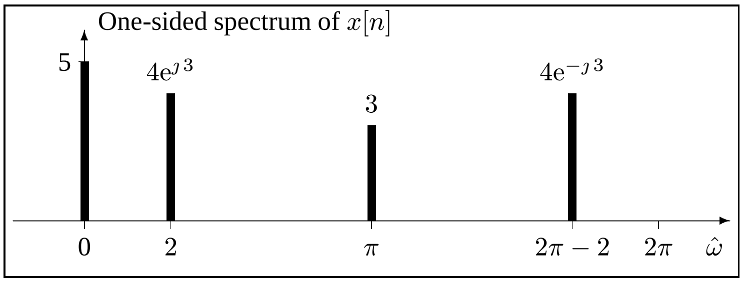
Example. Find the spectrum of the signal

$$x[n] = 5 + 3(-1)^n + 8\cos(2n + 3).$$

Before jumping into the DFT, we must ask: is $x[n]$ periodic? Here

$$x[n] = 5 + 3\cos(\pi n) + 8\cos(2\pi \frac{1}{\pi} n + 3).$$

Since $1/\pi$ is irrational, this signal is not periodic. So we cannot apply the DFT. But we can still determine the spectrum of $x[n]$.



Be sure to think about why the component at π is "3" rather than "3/2."

Equivalent frequencies revisited

The most important uses of equivalent frequencies are the following

$$e^{-\jmath \hat{\omega} n} = e^{\jmath (2\pi - \hat{\omega}) n} \qquad e^{-\jmath \frac{2\pi}{N} kn} = e^{\jmath \frac{2\pi}{N}(N-k)}.$$

Computing the DFT

There are three basic methods for "manually" determining the DFT of a signal:

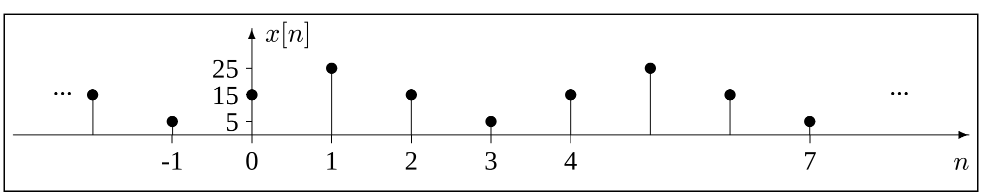
- using the DFT analysis formula,
- matching the DFT coefficients "by inspection,"
- combining the above with DFT properties.

The same techniques also work for the inverse DFT since it is almost the same formula!

Recall the N-point DFT formulas:

$$\text{Analysis: } X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} kn} \qquad \text{Synthesis: } x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N} kn}.$$

Example. Find the spectrum of the following signal. (Using the **analysis formula**.)



This signal is periodic. Its fundamental period is $N_0 = 4$, so we choose $N = N_0 = 4$ for the DFT.

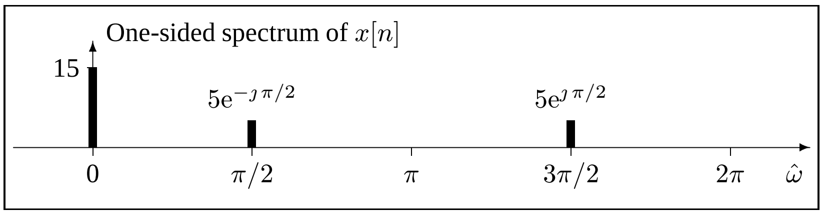
Applying the analysis formula:

$$
\begin{aligned}
X[k] &= \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-\jmath \frac{2\pi}{4}kn} = \frac{1}{4}\left[x[0]\, e^{-\jmath \frac{2\pi}{4}k0} + x[1]\, e^{-\jmath \frac{2\pi}{4}k1} + x[2]\, e^{-\jmath \frac{2\pi}{4}k2} + x[3]\, e^{-\jmath \frac{2\pi}{4}k3}\right] \\
&= \frac{1}{4}\left[15 + 25e^{-\jmath \frac{2\pi}{4}k} + 15e^{-\jmath \pi k} + 5e^{-\jmath \frac{2\pi}{4}k3}\right] = \begin{cases} \frac{1}{4}(15+25+15+5), & k=0 \\ \frac{1}{4}(15-25\jmath-15+5\jmath), & k=1 \\ \frac{1}{4}(15-25+15-5), & k=2 \\ \frac{1}{4}(15+25\jmath-15-5\jmath), & k=3 \end{cases} = \begin{cases} 15, & k=0 \\ -5\jmath, & k=1 \\ 0, & k=2 \\ 5\jmath, & k=3. \end{cases}
\end{aligned}
$$

The frequencies that correspond to these DFT coefficients are multiples of $2\pi/4$. So in list form the spectrum is

$$\left\{(15, 0)\,,\ \left(5e^{-\jmath \pi/2}, \pi/2\right),\ \left(5e^{\jmath \pi/2}, 3\pi/2\right)\right\}.$$

Graphically, the spectrum of this signal is the following.



It is also useful to be able to express $x[n]$ as a sum of complex exponentials or sinusoids. Applying the synthesis formula:

$$
\begin{aligned}
x[n] &= \sum_{k=0}^{3} X[k]\, e^{\jmath \frac{2\pi}{4}kn} = X[0]\, e^{\jmath \frac{2\pi}{4}0n} + X[1]\, e^{\jmath \frac{2\pi}{4}1n} + X[2]\, e^{\jmath \frac{2\pi}{4}2n} + X[3]\, e^{\jmath \frac{2\pi}{4}3n} \\
&= 15 + 5e^{-\jmath \pi/2} e^{\jmath \frac{2\pi}{4}n} + 5e^{-\pi/2} e^{\jmath \frac{2\pi}{4}3n} = 15 + 5e^{-\jmath \pi/2} e^{\jmath \frac{2\pi}{4}n} + 5e^{-\pi/2} e^{-\jmath \frac{2\pi}{4}n} = 15 + 10\cos(\frac{2\pi}{4}n - \frac{\pi}{2}),
\end{aligned}
$$

where we used the fact that $3\pi/2$ and $-\pi/2$ are equivalent frequencies. In this case, one could also recognize the final expression for $x[n]$ directly from its spectrum.

Example. (Using **coefficient matching**.)

Determine the 32-point DFT the following signal $x[n] = 20\sin^2(\frac{3\pi}{8}n)$.

The N-point DFT allows us to express a N-periodic signal as a sum of N complex exponentials:

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N}kn} = X[0] + X[1]\, e^{\jmath \frac{2\pi}{N}1n} + X[2]\, e^{\jmath \frac{2\pi}{N}2n} + \cdots + X[N-1]\, e^{\jmath \frac{2\pi}{N}(N-1)n}. \qquad \text{(3d-2)}$$

If we can find such an expression directly, then we do not need to use the analysis formula.
In this case, apply an inverse Euler identity:

$$
\begin{aligned}
x[n] &= 20\sin^2(\frac{3\pi}{8}n) = 20\left(\frac{e^{\jmath \frac{3\pi}{8}n} - e^{-\jmath \frac{3\pi}{8}n}}{2\jmath}\right)^2 \\
&= 20\left(\frac{e^{\jmath \frac{2\pi}{32}6n} - e^{-\jmath \frac{2\pi}{32}6n}}{2\jmath}\right)^2 = 5\left(2 - e^{\jmath \frac{2\pi}{32}12n} - e^{-\jmath \frac{2\pi}{32}12n}\right) \\
&= 10 - 5e^{\jmath \frac{2\pi}{32}12n} - 5e^{-\jmath \frac{2\pi}{32}12n} = 10 - 5e^{\jmath \frac{2\pi}{32}12n} - 5e^{\jmath \frac{2\pi}{32}20n},
\end{aligned}
$$

where in the last line we used the fact that $-\frac{2\pi}{32}12$ and $\frac{2\pi}{32}20$ are equivalent frequencies. Considering the final form, comparing to (3d-2) we see that the 32-point DFT of $x[n]$ is given by:

$$X[k] = \begin{cases} 10, & k=0 \\ -5, & k=12 \\ -5, & k=20 \\ 0, & \text{otherwise.} \end{cases}$$

We see that this signal has a DC term and two other complex exponential terms.

Example. (Using the **analysis formula**.)

Determine the 32-point DFT the following signal $y[n] = (1/5)^n$. Note that this is an infinite duration signal, so we are only computing the DFT of a segment of it for $0 \leq n \leq 31$. We chose $N = 32$ here simply for illustration.

Substitute $y[n]$ into the analysis formula and use a geometric series formula to help simplify:

$$\begin{aligned} Y[k] &= \frac{1}{32} \sum_{n=0}^{31} (1/5)^n \, e^{-\jmath \frac{2\pi}{32} kn} = \frac{1}{32} \sum_{n=0}^{31} \left((1/5) \, e^{-\jmath \frac{2\pi}{32} k} \right)^n \\ &= \frac{1}{32} \frac{1 - \left((1/5) \, e^{-\jmath \frac{2\pi}{32} k} \right)^{32}}{1 - (1/5) \, e^{-\jmath \frac{2\pi}{32} k}} = \frac{1}{32} \frac{1 - 1/5^{32}}{1 - (1/5) \, e^{-\jmath \frac{2\pi}{32} k}}. \end{aligned}$$

A plot of the magnitude spectrum shows that this signal has nearly the same power at all frequencies, with a bit more at the lower frequencies.

Example. (Using **properties**.)

Determine the 32-point DFT the following signal $z[n] = 40 \sin^2(\frac{3\pi}{8}(n-4)) + 7(1/5)^n$.

We see that $z[n] = 2x[n-4] + 7y[n]$. The **shift property** of the DFT is the following.

If $s[n] = x[n - n_0]$, then $S[k] = X[k] \, e^{-\jmath \frac{2\pi}{N} k n_0}$.

Thus, using the shift property and the **linearity** of the DFT, we see that

$$Z[k] = 2X[k] \, e^{-\jmath \frac{2\pi}{32} k 4} + 7Y[k] = \begin{cases} 20 + \frac{7}{32} \frac{1 - 1/5^{32}}{1 - (1/5)}, & k = 0 \\ -10 e^{-\jmath \frac{2\pi}{32} 4 \cdot 12} + \frac{7}{32} \frac{1 - 1/5^{32}}{1 - (1/5) \, e^{-\jmath \frac{2\pi}{32} 12}}, & k = 12 \\ -10 e^{-\jmath \frac{2\pi}{32} (4 \cdot 20)} + \frac{7}{32} \frac{1 - 1/5^{32}}{1 - (1/5) \, e^{-\jmath \frac{2\pi}{32} 20}}, & k = 20 \\ \frac{7}{32} \frac{1 - 1/5^{32}}{1 - (1/5) \, e^{-\jmath \frac{2\pi}{32} k}}, & \text{otherwise.} \end{cases}$$

So what?

After we know how to compute the DFT of a signal, what can we do? There are an enormous number of applications; the DFT and its fast computational version, the FFT, are the foundation for much of signal processing.

- As demonstrated in lecture, we can perform audio signal compression by discarding frequency components with small DFT coefficients. This is the essence of how MP3 works.
- In lab you will see how to use the DFT to remove a contaminating tone from an audio signal.
- If we start with a continuous-time signal and sample it to form a discrete-time signal and then compute the DFT of that discrete-time signal, then we will soon discuss how the DFT coefficients are related to the spectrum of the original continuous-time signal. This is how instruments like digital oscilloscopes display the (approximate) spectrum of continuous-time signals.

Example. (Using the **analysis formula**, a long example.)

Determine the 32-point DFT the following signal $x[n] = |\sin(\frac{3\pi}{8}n)|$. Note that the period of this signal is $N_0 = 8$, but there are still reasons why one might want to compute the N-point DFT for N larger than N_0; we chose $N = 32$ here simply for illustration.

Now substitute $x[n]$ into the analysis formula and use Euler to help simplify:

$$\begin{aligned}
X[k] &= \frac{1}{32}\sum_{n=0}^{31} |\sin(\frac{3\pi}{8}n)|\, e^{-\jmath\frac{2\pi}{32}kn} = \frac{1}{32}\left[\sum_{n=0}^{15}\sin(\frac{3\pi}{8}n)\, e^{-\jmath\frac{2\pi}{32}kn} - \sum_{n=16}^{31}\sin(\frac{3\pi}{8}n)\, e^{-\jmath\frac{2\pi}{32}kn}\right] \\
&= \frac{1}{32}\left[\sum_{n=0}^{15}\frac{e^{\jmath\frac{3\pi}{8}n} - e^{-\jmath\frac{3\pi}{8}n}}{2\jmath}\, e^{-\jmath\frac{2\pi}{32}kn} - \sum_{n=16}^{31}\frac{e^{\jmath\frac{3\pi}{8}n} - e^{-\jmath\frac{3\pi}{8}n}}{2\jmath}\, e^{-\jmath\frac{2\pi}{32}kn}\right] \\
&= \frac{1}{64\jmath}\left[\sum_{n=0}^{15}\left(e^{\jmath\frac{2\pi}{32}6n} - e^{-\jmath\frac{2\pi}{32}6n}\right) e^{-\jmath\frac{2\pi}{32}kn} - \sum_{n=16}^{31}\left(e^{\jmath\frac{2\pi}{32}6n} - e^{-\jmath\frac{2\pi}{32}6n}\right) e^{-\jmath\frac{2\pi}{32}kn}\right] \\
&= \frac{1}{64\jmath}\left[\sum_{n=0}^{15}\left(e^{\jmath\frac{2\pi}{32}(6-k)n} - e^{-\jmath\frac{2\pi}{32}(6+k)n}\right) - \sum_{n=16}^{31}\left(e^{\jmath\frac{2\pi}{32}(6-k)n} - e^{-\jmath\frac{2\pi}{32}(6+k)n}\right)\right] \\
&= \frac{1}{64\jmath}\left[\sum_{n=0}^{15} e^{\jmath\frac{2\pi}{32}(6-k)n} - \sum_{n=0}^{15} e^{\jmath\frac{2\pi}{32}(26-k)n} - \sum_{n=16}^{31} e^{\jmath\frac{2\pi}{32}(6-k)n} + \sum_{n=16}^{31} e^{\jmath\frac{2\pi}{32}(26-k)n}\right].
\end{aligned}$$

Apparently we will have to consider the cases $k = 6$ and $k = 26$ separately. For *other* values of k we apply a geometric series formula to find:

$$\begin{aligned}
X[k] &= \frac{1}{64\jmath}\left[\frac{1-\left(e^{\jmath\frac{2\pi}{32}(6-k)}\right)^{16}}{1-e^{\jmath\frac{2\pi}{32}(6-k)}} - \frac{1-\left(e^{\jmath\frac{2\pi}{32}(26-k)}\right)^{16}}{1-e^{\jmath\frac{2\pi}{32}(26-k)}} - (-1)^k\frac{1-(-1)^k}{1-e^{\jmath\frac{2\pi}{32}(6-k)}} + (-1)^k\frac{1-(-1)^k}{1-e^{\jmath\frac{2\pi}{32}(26-k)}}\right] \\
&= \frac{1-(-1)^k}{32\jmath}\left[\frac{1}{1-e^{\jmath\frac{2\pi}{32}(6-k)}} - \frac{1}{1-e^{\jmath\frac{2\pi}{32}(26-k)}}\right] = \frac{1-(-1)^k}{32\jmath}\cdot\frac{1-e^{\jmath\frac{2\pi}{32}(26-k)} - [1-e^{\jmath\frac{2\pi}{32}(6-k)}]}{[1-e^{\jmath\frac{2\pi}{32}(6-k)}][1-e^{\jmath\frac{2\pi}{32}(26-k)}]} \\
&= \frac{1-(-1)^k}{32\jmath}\,\frac{e^{\jmath\frac{2\pi}{32}(6-k)} - e^{\jmath\frac{2\pi}{32}(-6-k)}}{[1-e^{\jmath\frac{2\pi}{32}(6-k)}][1-e^{\jmath\frac{2\pi}{32}(-6-k)}]} \\
&= \frac{1-(-1)^k}{16}\,\frac{e^{-\jmath\frac{2\pi}{32}k}\sin(\frac{2\pi}{32}6)}{1-e^{\jmath\frac{2\pi}{32}(6-k)} - e^{\jmath\frac{2\pi}{32}(-6-k)} + e^{\jmath\frac{2\pi}{32}(-2k)}} \\
&= \frac{1-(-1)^k}{16}\,\frac{\sin(\frac{2\pi}{32}6)}{e^{\jmath\frac{2\pi}{32}k} - e^{\jmath\frac{2\pi}{32}6} - e^{\jmath\frac{2\pi}{32}(-6)} + e^{\jmath\frac{2\pi}{32}(-k)}} \\
&= \frac{1-(-1)^k}{8}\,\frac{\sin(\frac{3\pi}{8})}{\cos(\frac{2\pi}{32}k) - \cos(\frac{3\pi}{8})}.
\end{aligned}$$

This is too messy to serve as a classroom example. However, the original signal $x[n]$ is real and even, so the above manipulations show that with enough simplification, we can indeed find DFT coefficients that are real.

For $k = 6$ we have

$$X[6] = \frac{1}{64\jmath}\left[\sum_{n=0}^{15} 1 - \sum_{n=0}^{15} e^{\jmath\frac{2\pi}{32}20n} - \sum_{n=16}^{31} 1 + \sum_{n=16}^{31} e^{\jmath\frac{2\pi}{32}20n}\right] = 0$$

and similarly $X[26] = 0$. So our final answer is

$$X[k] = \begin{cases} 0, & k = 6, 26 \\ \dfrac{1-(-1)^k}{8}\,\dfrac{\sin(\frac{3\pi}{8})}{\cos(\frac{2\pi}{32}k) - \cos(\frac{3\pi}{8})} & \text{otherwise.} \end{cases} = \begin{cases} 0, & k \text{ even} \\ \dfrac{1}{4}\,\dfrac{\sin(\frac{3\pi}{8})}{\cos(\frac{2\pi}{32}k) - \cos(\frac{3\pi}{8})} & k \text{ odd.} \end{cases}$$

Admittedly this problem would be easier to solve (approximately) using MATLAB.

D.3 Derivation of the DFT

To demonstrate the validity of the theorem, we will first show that when the analysis formula for $X[k]$ is substituted into the synthesis formula, the result is $x[n]$. We will then show that when the synthesis formula holds, the analysis formula is the one and only way to determine the coefficients.

These demonstrations rely on the following fact, which we will derive before demonstrating the validity of the DFT Theorem. This fact will also be useful at other times in the course.

Fact D1

$$\sum_{n=0}^{N-1} e^{\jmath \frac{2\pi}{N}(k-m)n} = \begin{cases} N, & k = m,\ k = m \pm N,\ k = m \pm 2N, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Derivation.

- Case 1. When $k = m + lN$ for $l \in \mathbb{Z}$, the exponent of each term is 0 or an integer multiple of 2π, making each term equal 1. Hence, in this case $\sum_{n=0}^{N-1} e^{\jmath \frac{2\pi}{N}(k-m)n} = N$.
- Case 2. When $k \neq m + lN$ for every integer l, to simplify notation, let $z = e^{\jmath \frac{2\pi}{N}(k-m)}$. Notice that $z \neq 1$ in this case. Using this definition and applying the finite geometric series formula:

$$\sum_{n=0}^{N-1} e^{\jmath \frac{2\pi}{N}(k-m)n} = \sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z} = \frac{1-e^{\jmath \frac{2\pi}{N}(k-m)N}}{1-z} = \frac{1-e^{\jmath 2\pi(k-m)}}{1-z} = \frac{1-1}{1-z} = 0,$$

  since the exponent in numerator is $\jmath$ times a multiple of 2π.

Derivation of the DFT Theorem

Substituting the analysis formula for $X[k]$ into the synthesis formulas gives

$$\begin{aligned} \sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N}kn} &= \sum_{k=0}^{N-1} \left[\frac{1}{N} \sum_{n'=0}^{N-1} x[n']\, e^{-\jmath \frac{2\pi}{N}kn'} \right] e^{\jmath \frac{2\pi}{N}kn} \\ &\qquad \text{using } n' \text{ in analysis formula since } n \text{ already used in synthesis formula} \\ &= \frac{1}{N} \sum_{n'=0}^{N-1} x[n'] \left[\sum_{k=0}^{N-1} e^{\jmath \frac{2\pi}{N}k(n-n')} \right] \qquad \text{rearranging terms} \\ &= \frac{1}{N} x[n]\, N = x[n] \qquad \text{if } n = 0, \ldots, N-1, \text{ using Fact D1.} \end{aligned}$$

Specifically, the rightmost sum equals 0 when the exponent is nonzero, *i.e.*, when $n' \neq n$, and equals N when the exponent is zero, *i.e.*, when $n' = n$. Therefore, $x[n']$ is multiplied by 0 when $n' \neq n$, and by N when $n' = n$.

So, we have just shown that if we apply the analysis formula to any signal $x[n]$, then we when apply the synthesis formula to the resulting $X[k]$'s, we will get back the $x[n]$'s that we started with for $n = 0, \ldots, N-1$.

But could there be *other* $X[k]$'s that also could be used to synthesize the signal $x[n]$? The answer is "no" as shown in the next argument.

Finally, we show that if the synthesis formula holds, the coefficients must be calculated via the analysis formula. Suppose we have some $X[k]$'s for which the synthesis formula holds, *i.e.*,

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N}kn}, \qquad n = 0, \ldots, N-1.$$

We want to show that these $X[k]$'s must be those that come from the analysis formula.

We correlate both sides of this equation with $e^{\jmath \frac{2\pi}{N}k'n}$. That is, we first multiply both sides of the above synthesis formula by $\left(e^{\jmath \frac{2\pi}{N}k'n}\right)^\star = e^{-\jmath \frac{2\pi}{N}k'n}$ as follows

$$x[n]\, e^{-\jmath \frac{2\pi}{N}k'n} = \left[\sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N}kn} \right] e^{-\jmath \frac{2\pi}{N}k'n}.$$

Since this equality holds for $n = 0, \ldots, N-1$, we now sum over all these values of n:

$$\begin{aligned}
\sum_{n=0}^{N-1} x[n]\, \mathrm{e}^{-\jmath \frac{2\pi}{N} k' n} &= \sum_{n=0}^{N-1} \left[\sum_{k=0}^{N-1} X[k]\, \mathrm{e}^{\jmath \frac{2\pi}{N} kn} \right] \mathrm{e}^{-\jmath \frac{2\pi}{N} k' n} \\
&= \sum_{k=0}^{N-1} X[k] \left[\sum_{n=0}^{N-1} \mathrm{e}^{\jmath \frac{2\pi}{N} (k-k') n} \right] \qquad \text{rearranging terms} \\
&= X[k']\, N,
\end{aligned}$$

where the last equality is due to Fact D1 again. Specifically, the rightmost sum equals 0 when the exponent is not zero, *i.e.*, when $k \neq k'$, and equals N when the exponent is zero, *i.e.*, when $k = k'$. Therefore, $X[k]$ is multiplied by 0 when $k \neq k'$, and by N when $k = k'$. Rearranging the last equality and using k in place of k' yields

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, \mathrm{e}^{-\jmath \frac{2\pi}{N} kn},$$

which is the analysis formula given in the DFT Theorem.

D.4 Properties of the DFT

This section lists a number of useful properties of the DFT.

D1. Uniqueness
There is a one-to-one relationship between periodic signals with period N and sets of DFT coefficients. Specifically, for any given signal $x[n]$, the analysis formula gives the unique set of coefficients from which the synthesis formula yields $x[n]$.

This implies that the DFT coefficients can sometimes be found by means other than the analysis formula, *e.g.*, inspection. That is, if by some means you find $X[k]$ such that

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N} kn},$$

then this $X[k]$ is necessarily the DFT that would be computed by the analysis formula.

Similarly, for any given set of DFT coefficients $X[k]$, the synthesis formula gives the unique signal $x[n]$ with period N from which the analysis formula yields $X[k]$. That is, if by some means you find a signal $x[n]$ such that

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} kn},$$

then that signal $x[n]$ is the one and only signal having $X[k]$ as its coefficients.

Another statement of the one-to-oneness is that if $x_1[n]$ and $x_2[n]$ are distinct periodic signals, *i.e.*, $x_1[n] \neq x_2[n]$ for at least one value of n, each with period N, then for at least one k, $X_1[k] \neq X_2[k]$.

D2. Mean value
$X[0]$ is the mean or DC value of $x[n]$.

This is because

$$X[0] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} 0n} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] = \mathrm{M}(x)\,,$$

D3. Summation interval
For a N-periodic signal $x[n]$, one can compute the DFT coefficients by summing over any time interval of length N.

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} kn} = \frac{1}{N} \sum_{n=m}^{m+N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} kn} \qquad \forall m \in \mathbb{Z}.$$

D4. Conjugate symmetry (important)
When the signal $x[n]$ is real,

$$X[N-k] = X^*[k]\,, \qquad k = 1, \ldots, N-1$$

and $X[0] = X^*[0]$ so $X[0]$ is real.

This shows that if one knows $X[k]$ for $k = 0, \ldots, N/2$, then one can easily find the remaining $X[k]$'s. This "redundancy" is often exploited in digital implementations to reduce memory requirements.

Note that $X[N-k]$ is the spectral component at frequency $\frac{2\pi}{N}(N-k)$, which is equivalent to frequency $-\frac{2\pi}{N}k$.

Derivation. (This property does not apply to complex signals!)

$$\begin{aligned}
X^*[k] &= \left[\frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} kn}\right]^\star = \frac{1}{N} \sum_{n=0}^{N-1} x^*[n]\, e^{\jmath \frac{2\pi}{N} kn} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{\jmath \frac{2\pi}{N} kn} \text{ because } x[n] \text{ is real so } x^*[n] = x[n] \\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} (N-k)n} \text{ because } \tfrac{2\pi}{N}k \text{ and } -\tfrac{2\pi}{N}(N-k) \text{ are equivalent frequencies} \\
&= X[N-k]\,.
\end{aligned}$$

D5. Sinusoids (important)
Conjugate pairs of coefficients synthesize a sinusoid When the signal $x[n]$ is real,

$$\begin{aligned} X[k]\,e^{\jmath\,\frac{2\pi}{N}kn} + X[N-k]\,e^{\jmath\,\frac{2\pi}{N}(N-k)n} &= X[k]\,e^{\jmath\,\frac{2\pi}{N}kn} + X^*[k]\,e^{-\jmath\,\frac{2\pi}{N}kn} \\ &= 2|X[k]|\cos(\frac{2\pi}{N}kn + \angle X[k]). \end{aligned}$$

Thus, when looking at a spectrum, one should "see" sinusoidal signal components, one for each conjugate pair of coefficients.

D6. Linearity
Suppose $x[n]$ and $y[n]$ are periodic with period N and have $X[k]$ and $Y[k]$ as their N-point DFTs, respectively. Then the N-point DFT of $s[n] = \alpha x[n] + \beta y[n]$ is $S[k] = \alpha X[k] + \beta Y[k]$.

Similarly, if $X[k]$ and $Y[k]$ are sequences of length N with inverse DFTs given by $x[n]$ and $y[n]$, then the inverse DFT of $\alpha X[k] + \beta Y[k]$ is $\alpha x[n] + \beta y[n]$.

The derivations of these properties are left as exercises.

D7. Parseval's theorem
For a real or complex signal $x[n]$ that is periodic with period N, we can computer the average power in the time domain or in the frequency domain as follows:

$$\text{signal average power} = \text{MS}(x) = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |X[k]|^2.$$

Recall that the average power of a N-periodic signal $x[n]$ is

$$\text{MS}(x) = \lim_{M\to\infty} = \frac{1}{2M+1}\sum_{n=-M}^{M} |x[n]|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2.$$

Derivation.

$$\begin{aligned} \text{MS}(x) &= \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\,x^*[n] \\ &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\left[\sum_{k=0}^{N-1} X[k]\,e^{\jmath\,\frac{2\pi}{N}kn}\right]^\star \qquad \text{synthesis formula} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\left[\sum_{k=0}^{N-1} X^*[k]\,e^{-\jmath\,\frac{2\pi}{N}kn}\right] \\ &= \sum_{k=0}^{N-1} X^*[k]\left[\frac{1}{N}\sum_{n=0}^{N-1} x[n]\,e^{-\jmath\,\frac{2\pi}{N}kn}\right] \qquad \text{rearranging} \\ &= \sum_{k=0}^{N-1} X^*[k]\,X[k] \qquad \text{by the analysis formula} \\ &= \sum_{k=0}^{N-1} |X[k]|^2. \end{aligned}$$

Example. (The **impulse train** signal.)

$$x[n] = \begin{cases} A, & n = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise.} \end{cases} \qquad \textbf{\textit{(Picture)}}$$

$$X[k] = \frac{1}{N}\sum_{k=0}^{N-1} x[n]\,e^{-\jmath\,\frac{2\pi}{N}kn} = \frac{1}{N}A = \frac{A}{N} \qquad \textbf{\textit{(Picture)}}\ .$$

Time domain: $\text{MS}(x) = \frac{1}{N}A^2$. Frequency domain: $\text{MS}(x) = \sum_{k=0}^{N-1} |X[k]|^2 = \sum_{k=0}^{N-1}\left(\frac{A}{N}\right)^2 = N\left(\frac{A}{N}\right)^2 = A^2/N$.

The following properties will be emphasized less in this class.

D8. Choice of period

Suppose $x[n]$ is periodic with period N, suppose $X[k]$ is the N-point DFT of $x[n]$, and suppose $X'[k]$ is the 2N-point DFT of $x[n]$. Then,

$$X'[k] = \begin{cases} X[k/2], & k = 0, 2, 4, \ldots, 2N-2 \\ 0, & \text{otherwise.} \end{cases}$$

This means that the (one-sided) spectrum based on the $2N$-point DFT is the same as that based on the N-point DFT. For example if N is even, the spectrum based on the $2N$-point DFT is

$$\left\{ (X'[0], 0), \left(X'[2], \frac{2\pi}{N} 2 \right), \ldots \left(X'[2N-2], \frac{2\pi}{N}(2N-2) \right) \right\}$$
$$= \left\{ (X[0], 0), \left(X[1], \frac{2\pi}{N} 1 \right), \ldots \left(X[N-1], \frac{2\pi}{N}(N-1) \right) \right\},$$

which is the one-sided spectrum based on the N-point DFT.

The derivation is left as an exercise.

D9. Time shift

If $x[n]$ has DFT $X[k]$, then $s[n] = x[n - n_0]$ has DFT coefficients

$$S[k] = X[k] \, e^{-\jmath \frac{2\pi}{N} k n_0}.$$

This shows, not surprisingly, that a time shift causes a phase shift of each spectral component, where the phase shift is proportional to the frequency of the component. The derivation is left as an exercise.

D10. Frequency shift

If $x[n]$ has DFT $X[k]$, then $y[n] = x[n] \, e^{\jmath \frac{2\pi}{N} m n}$ has DFT coefficients[1]

$$Y[k] = X[k - m].$$

This shows that multiplying a signal by a complex exponential has the effect of shifting the spectrum of the signal. The derivation is left as an exercise.

D11. Time scaling

This is not as straightforward as in the continuous-time case and will not be discussed here, except to indicate that if m is a positive integer, then $y[n] = x[nm]$ is a **subsampled** version of $x[n]$, which is well defined. (In contrast, the expression $x[n/m]$ is not defined for all values of n.) See EECS 451 for thorough coverage.

D12. Finite sums?

Since the DFT synthesis formula is a finite sum, we can compute the values “exactly” (to within the precision of our computers), unlike in the case of continuous-time signals where we usually have to make finite approximations to the infinite sums.

However, for data compression problems such as MP3 audio coding, we can save memory by discarding small DFT coefficients thereby reducing the finite sum to an even smaller number of terms, at the expense of imperfect signal synthesis.

D13. Technicalities?

Since the sums in the synthesis and analysis formulas are finite, no technical conditions are needed as are required for the Fourier series.

D14. Real and even signals

If $x[n]$ is real and even ($x[-n] = x[n]$), then the DFT coefficients are real.

Example. ***(Picture)*** $y[n] = 24x[n] - 16x[n-4]$ where $x[n]$ is the impulse train worked out earlier with $A = 1$ and $N = 8$.

$Y[k] = 24X[k] - 16X[k] \, e^{-\jmath \frac{2\pi}{8}[4]k} = 24\frac{1}{8} - 16\frac{1}{8}(-1)^k = 3 - 2(-1)^k = \begin{cases} 1, & k \text{ even} \\ 5, & k \text{ odd} \end{cases}$ ***(Picture)***

So $x[n] = 1 + 10\cos(\frac{\pi}{4}n) + 2\cos(\frac{\pi}{2}n) + 10\cos(\frac{3\pi}{4}n) + \cos(\pi n)$.

[1] There is a subtle technicality here; the expression $k - m$ must be interpreted modulo N.

Two-sided spectra and the DFT

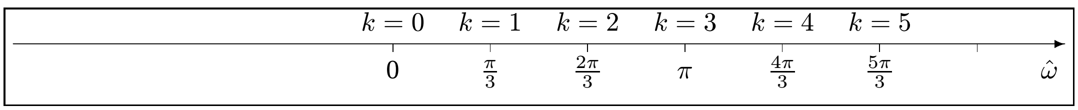
What if we are interested in a **two-sided spectrum** instead of a one-sided spectrum? Then we need to examine each frequency in the list (3d-1) in the DFT theorem above and adjust by 2π as necessary to find equivalent frequencies that are in the range $[-\pi, \pi)$. For most of the frequencies it is easy to see what is needed. However, for the frequencies at or near π, we find that the necessary adjustment depends on whether N is even or odd. Working through the details, one finds that the two-sided spectrum of a periodic signal with period N is concentrated at the following frequencies:

$$-\pi,\ -\frac{2\pi}{N}\left(\frac{N}{2}-1\right),\ \ldots,\ -\frac{2\pi}{N}2,\ -\frac{2\pi}{N}1,\ 0,\ \frac{2\pi}{N}1,\ \frac{2\pi}{N}2,\ \ldots,\ \frac{2\pi}{N}\left(\frac{N}{2}-1\right) \qquad \text{when } N \text{ is even}$$

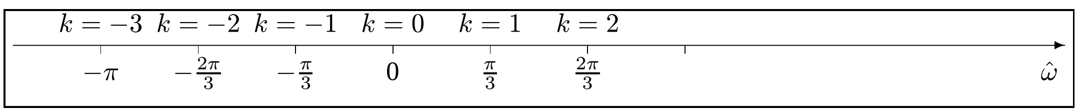
$$-\frac{2\pi}{N}\left(\frac{N-1}{2}\right),\ \ldots,\ -\frac{2\pi}{N}2,\ -\frac{2\pi}{N}1,\ 0,\ \frac{2\pi}{N}1,\ \frac{2\pi}{N}2,\ \ldots,\ \frac{2\pi}{N}\left(\frac{N-1}{2}\right) \qquad \text{when } N \text{ is odd.}$$

It is a bit inconvenient that the two-sided spectrum depends on whether N is even or odd, and this "messiness" is probably why the DFT is usually defined so as to directly give a one-sided spectrum. We will mostly use the one-sided spectrum. However, for relating discrete-time and continuous-time spectra, we need two-sided spectra, so we illustrate some two-sided spectra using examples.

Example. If $N = 6$, which is even, then the following figure shows the 6 **DFT frequencies** for a one-sided spectrum.



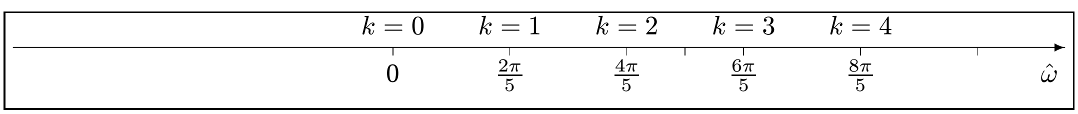
The following figure shows the frequencies for a two-sided spectrum for $N = 6$.



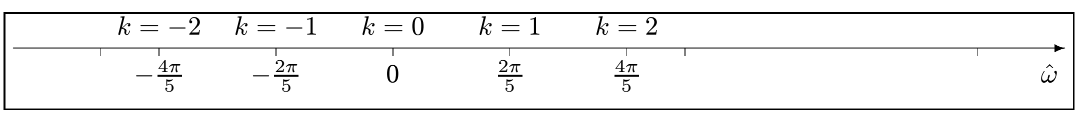
Notice that there is no component at π since $-\pi$ and π are equivalent frequencies so there is no need for both. This makes the spectrum appear somehow "asymmetric" for even values of N.

Our decision to use frequencies over $[-\pi, \pi)$ rather than over $(-\pi, \pi]$ is arbitrary, and not a universal convention. However, the choice given here is consistent with MATLAB's `fftshift` command.

Example. If $N = 5$, which is odd, then the following figure shows the 5 DFT frequencies for a one-sided spectrum.



The following figure shows the frequencies for the $N = 5$ two-sided spectrum.



The odd case looks more symmetric than the even case. However, even values of N are used more frequently in practice because the FFT is fastest when N is a power of 2.

In summary, when N is even, the two-sided spectrum is

$$\left\{ \left(X\left[\frac{N}{2}\right], -\pi\right),\ \left(X\left[\frac{N}{2}+1\right], -\frac{2\pi}{N}\left(\frac{N}{2}-1\right)\right),\ \ldots,\ \left(X[N-1], -\frac{2\pi}{N}1\right), \right.$$
$$\left. (X[0], 0),\ \left(X[1], \frac{2\pi}{N}1\right),\ \ldots,\ \left(X\left[\frac{N}{2}-1\right], \frac{2\pi}{N}\left(\frac{N}{2}-1\right)\right) \right\},$$

and when N is odd the two-sided spectrum is the following

$$\left\{ \left(X\left[\frac{N+1}{2}\right], -\frac{2\pi}{N}\left(\frac{N-1}{2}\right)\right),\ \left(X\left[\frac{N+3}{2}\right], -\frac{2\pi}{N}\left(\frac{N-3}{2}\right)\right),\ \ldots,\ \left(X[N-2], -\frac{2\pi}{N}2\right),\ \left(X[N-1], -\frac{2\pi}{N}1\right), \right.$$
$$\left. (X[0], 0)\ \left(X[1], \frac{2\pi}{N}1\right),\ \left(X[2], \frac{2\pi}{N}2\right),\ \ldots,\ \left(X\left[\frac{N-1}{2}\right], \frac{2\pi}{N}\left(\frac{N-1}{2}\right)\right) \right\}.$$

E. The spectra of segments of a signal

Question: How can we assess the spectrum of a signal that is not periodic?

For example, what if the signal has finite support? Or what if the signal has infinite support, but is not periodic?

Observation: The DFT analysis formula uses only a finite segment of a signal.

Signals with finite support

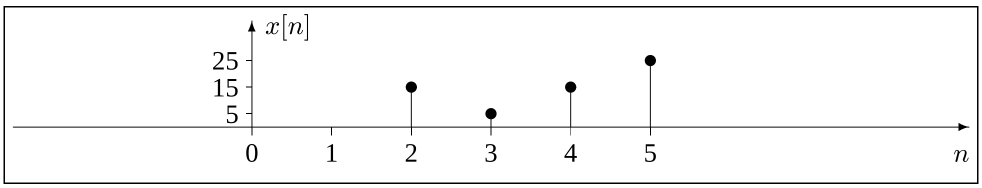
To assess the spectrum of a signal $x[n]$ with finite support $\{n_1, \ldots, n_2\}$, we can apply the DFT analysis formula directly to the signal over its support interval.

Let us begin by defining $\tilde{x}(t)$ to be a periodic signal that equals $x[n]$ on the interval $\{n_1, \ldots, n_2\}$ and simply repeats this behavior on other intervals of the same length. That is, let $N = n_2 - n_1 + 1$, and let

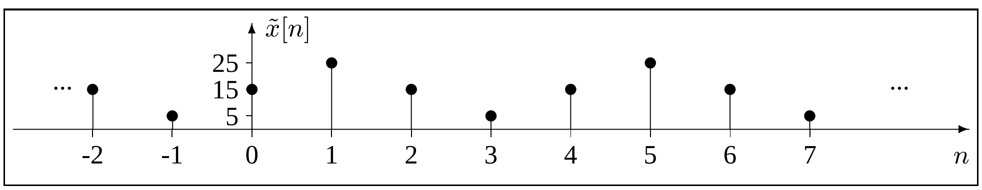
$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n - mN] \,.$$

The signal $\tilde{x}[n]$ is called the **periodic extension** of $x[n]$. Its period N is the support length of $x[n]$.

Example. Here is a signal with finite support.



Here is its periodic extension.



Returning to the general case, taking the N-point DFT of $\tilde{x}[n]$, we find

$$\tilde{x}[n] = \sum_{k=0}^{N-1} X[k] \, e^{\jmath \frac{2\pi}{N} kn} \qquad \text{(synthesis formula)}$$

where

$$X[k] = \frac{1}{N} \sum_{n=n_1}^{n_2} \tilde{x}[n] \, e^{-\jmath \frac{2\pi}{N} kn} \qquad \text{(analysis formula),}$$

and where we have used the fact that the analysis formula can have summation limits that cover any interval of length N.

Now we note that since

$$x[n] = \tilde{x}[n] \ \text{ for } n_1 \leq n \leq n_2,$$

we also have

$$x[n] = \sum_{k=0}^{N-1} X[k] \, e^{\jmath \frac{2\pi}{N} kn}, \qquad n_1 \leq n \leq n_2, \qquad \text{(synthesis formula)}$$

$$X[k] = \frac{1}{N} \sum_{n=n_1}^{n_2} x[n] \, e^{-\jmath \frac{2\pi}{N} kn} \qquad \text{(analysis formula).}$$

Thus we may view the two formulas above as synthesis and analysis formulas for a spectral representation of $x[n]$. The synthesis formula shows that, on its support interval, $x[n]$ can be viewed as the sum of complex exponentials with frequencies that are multiples of $2\pi/N$. The analysis formula shows how to find the spectral components. It is important to note that the synthesis formula yields $x[n]$ only in the support interval. Outside the support interval it yields $\tilde{x}[n]$, rather than $x[n] = 0$.

In summary, for a signal with finite support, we take the (one-sided) spectrum to be

$$\left\{ (X[0]\,, 0)\,,\ \left(X[1]\,, \frac{2\pi}{N} 1 \right),\ \left(X[2]\,, \frac{2\pi}{N} 2 \right),\ \ldots,\ \left(X[N-1]\,, \frac{2\pi}{N} (N-1) \right). \right\}$$

just as we did for periodic signals.

Note. Though we have introduced the DFT as fundamentally applying to periodic signals and secondarily applying to signals with finite support, some people take the opposite point of view, which is also valid.

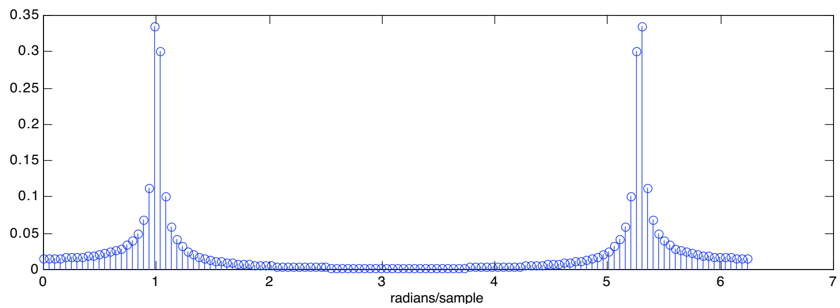
Aperiodic signals with infinite support

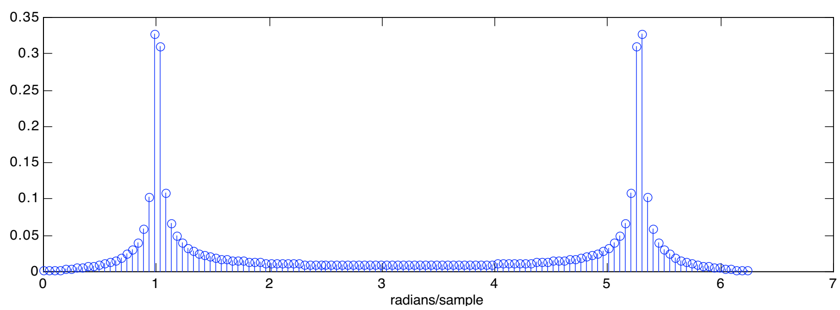
A common approach to assessing the spectrum of an aperiodic signal with infinite support is to choose an integer N, divide the time axis into segments $[0, N-1]$, $[N, 2N-1]$, $[2N, 3N-1]$, etc, and apply the above DFT approach to each segment. This yields a sequence of spectra, one for each segment.

Since the signal is not periodic, the data within each segment will be different. Thus the spectrum will differ from segment to segment. For example, the spectrum of the signal

$$x[n] = \cos(3.2n)$$

is shown below for two different segments of length $N = 128$.





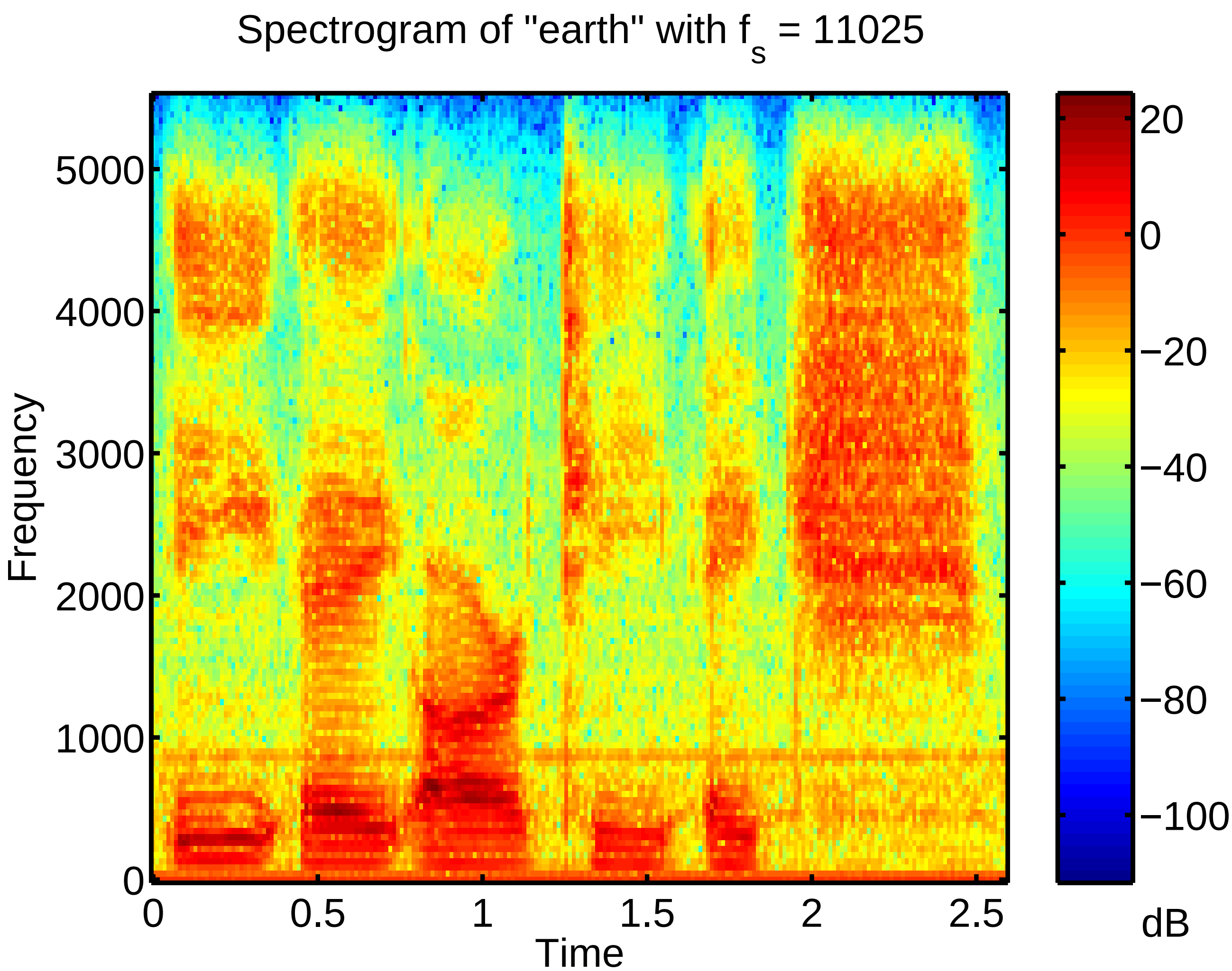
Notice that these two spectra are quite similar. Notice also that even though the signal is a pure sinusoidal signal, whose spectrum, according to the discussion of Section C, is concentrated entirely at frequencies 3.2 and -3.2, the spectra above show a couple of strong components in the vicinity of 3.2 and -3.2, and small components at other harmonic frequencies. This may be viewed as being due to the fact that we are using harmonic frequencies to synthesize a sinusoid whose frequency is not harmonic. It may also be viewed as being due to the fact that these spectra are actually the spectra of a periodic extension $\tilde{x}[n]$ of $x[n]$. A more thorough discussion, which would derive the actual form of the spectra shown above, is left to future courses.

The fact that we now have two different ways of assessing the spectrum of signals such as $x[n] = \cos(3.2n)$, as in Section C and as discussed here, may seem somewhat disconcerting. But this is reflective of the fact that, as mentioned earlier, the **spectrum** is a broad concept, like "economy" or "health," that has no simple, universal definition.

Spectrograms

Let us also mention that there are aperiodic signals for which it makes very good sense that the spectrum should differ from segment to segment. For example, the signal produced by musical instrument can be viewed as having a spectrum that changes with each note. This and other examples can be found in Section 3.5 of the text and in the Demos on the CD ROM relating to Chapter 3.

For such signals, it is common practice to apply the DFT in a sliding fashion. That is, the DFT is applied successively to overlapping intervals $[0, N-1]$, $[M, M+N-1]$, $[2M, N+2M-1]$, and so on, where $M \leq N$. A **spectrogram** is a plot showing the magnitudes of the DFT coefficients for each interval (usually by representing the magnitude as a color or graylevel) plotted over the starting time of the interval. For example a spectrogram of someone speaking the five letters 'e', 'a', 'r', 't', 'h' is shown below[2].



[2]The colors in this plot are visible when the pdf file is displayed on a color monitor.
The axes have been labelled in terms of the continuous-time values using the methods described in the next section.

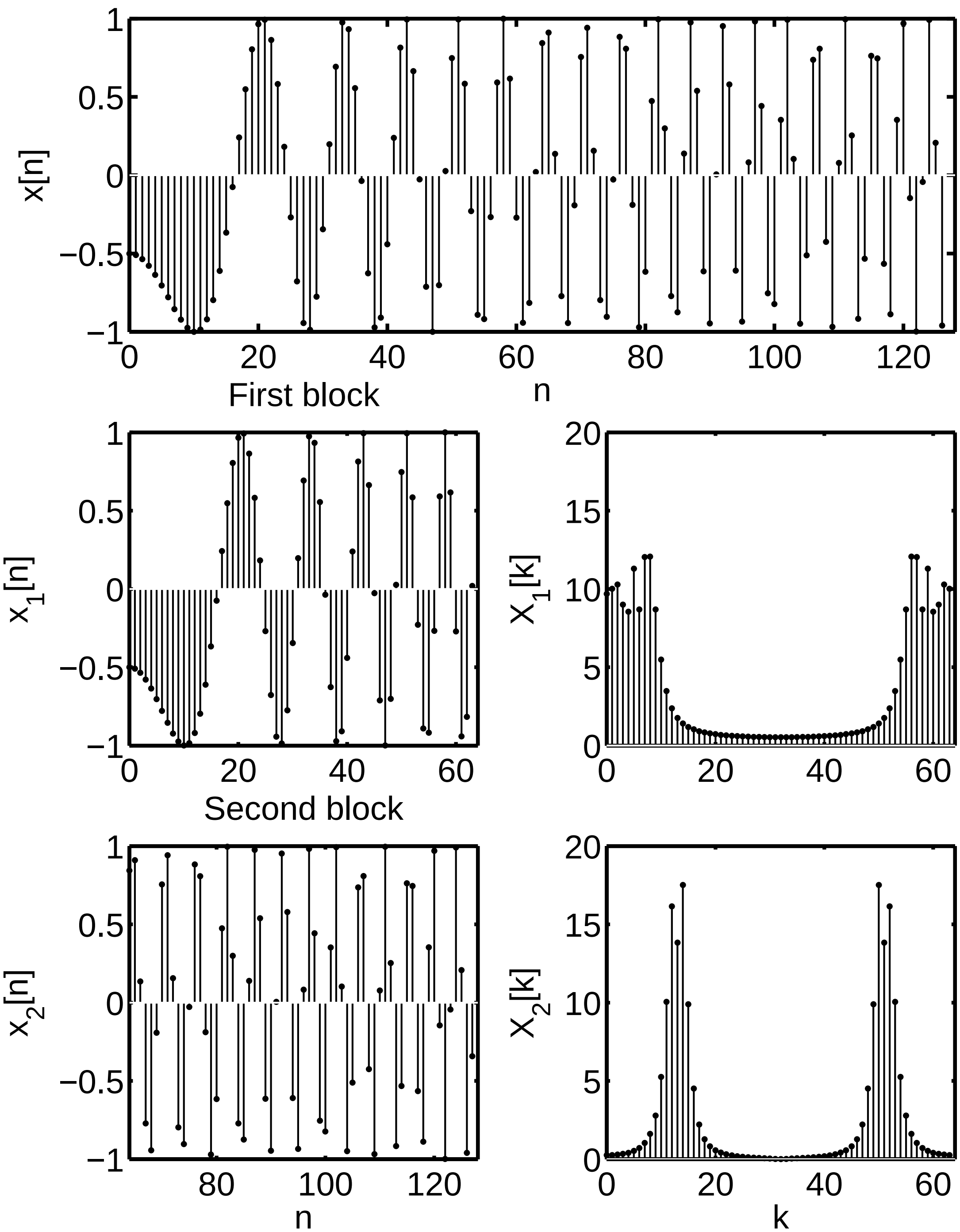
Example

Here is another aperiodic signal

$$x[n] = \cos(\frac{2\pi}{3}(n + 100e^{-n/100})).$$

We compute the DFT over the 1st block of 64 samples and then over the 2nd block of 64 samples.

It is clearly from the DFT coefficients that the second block has more of its power at higher frequencies, as expected in this case from the nature of the signal.



F. The relationship between the spectrum of a continuous-time signal and that of its samples

Frequently, we are often interested in the spectrum of some continuous-time signal, but for practical reasons, we sample the signal and work with the resulting discrete-time signal. If possible, we would like to be able to deduce the spectrum of the continuous-time signal from that of the discrete-time signal. This section shows how this can be done, at least approximately.

Example. We first illustrate the idea with the simplest possible example: a continuous-time sinusoidal signal.

$$x(t) = \cos(2\pi f_0 t) \to \boxed{\text{Sample } T_\text{s}} \to x[n] = x(nT_\text{s})$$

Using the sampling relationship:

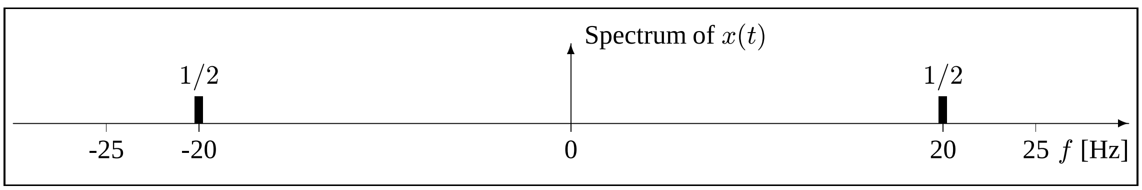
$$x[n] = x(nT_\text{s}) = \cos(2\pi f_0 (nT_\text{s})) = \cos(2\pi f_0 T_\text{s} n)$$

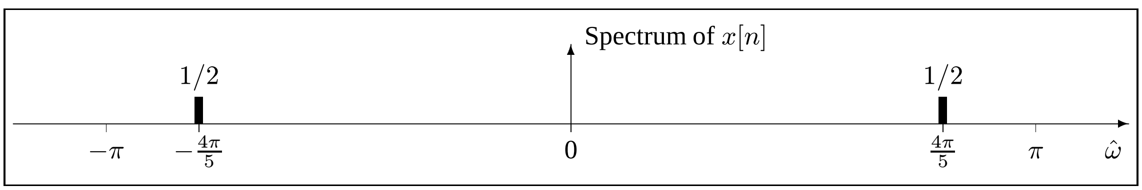
so we see that a continuous-time sinusoid of frequency f_0 becomes, after sampling, a discrete-time sinusoid of frequency

$$\boxed{\hat{\omega}_0 = 2\pi f_0 T_\text{s} = 2\pi \frac{f_0}{f_\text{s}}.} \tag{3d-3}$$

Example. Suppose $f_0 = 20$Hz, and $T_\text{s} = 0.02$s so $f_\text{s} = 50$Hz. And $\hat{\omega}_0 = 2\pi 20/50 = 4\pi/5$.

Then the spectra of $x(t)$ and $x[n]$ are as follows.





What if we do not have an equation for the signal?

In the preceding example, we have an analytical expression for $x(t)$ so we can draw "theoretical" spectra. In practice, if we have a "mystery signal" $x(t)$ whose spectrum we would like to examine, we must take a finite number, N, of samples of that signal, compute the DFT $X[k]$ of those samples, and then somehow relate the $X[k]$'s to the spectrum original signal $x(t)$.

The nature of what that DFT-based spectrum will look like will depend somewhat on the value of N that is chosen.

The following figure shows the DFT coefficients for various choices for N for the preceding example.

Consider first the choice $N = 40$, which is a multiple of the fundamental period of $x[n]$ which is $N_0 = 5$ in the preceding example. Then

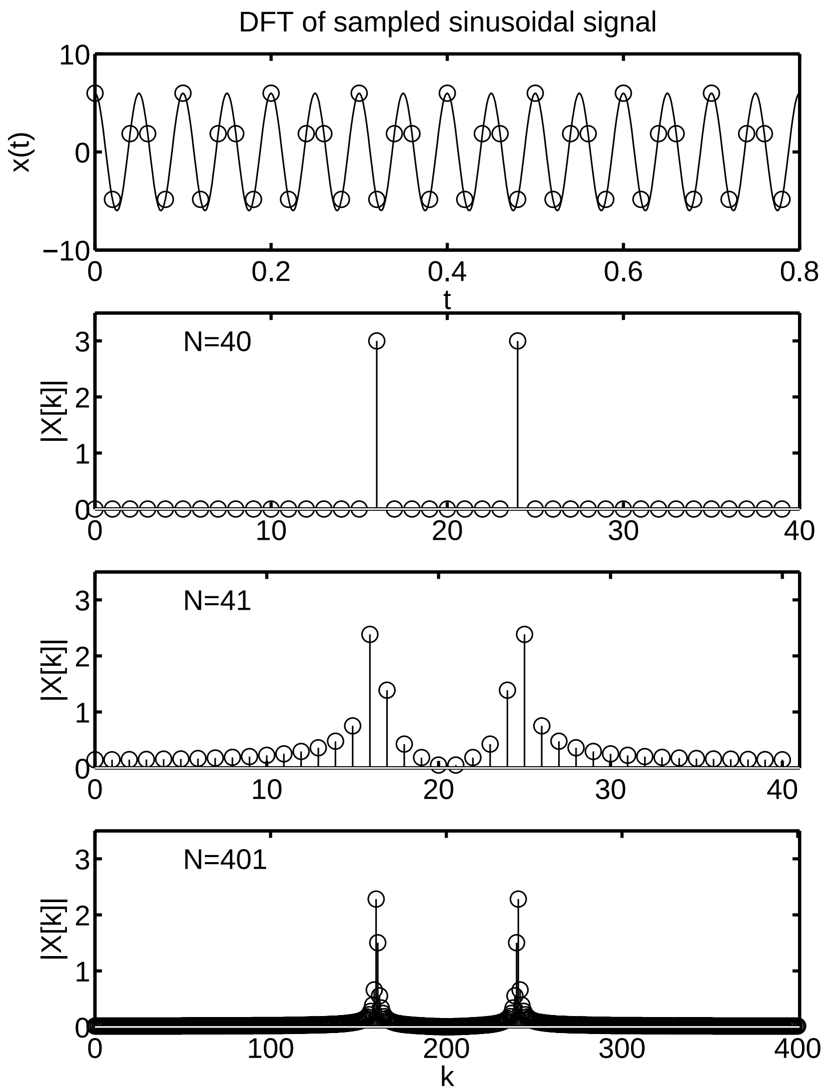
$$x[n] = \cos(2\pi \frac{2}{5} n) = \frac{1}{2} e^{\jmath \frac{2\pi}{5} 2n} + \frac{1}{2} e^{-\jmath \frac{2\pi}{5} 2n} = \frac{1}{2} e^{\jmath \frac{2\pi}{40} 16n} + \frac{1}{2} e^{-\jmath \frac{2\pi}{16} 40n}$$

so by coefficient matching we see that the 40-point DFT of $x[n]$ is given by

$$X[k] = \begin{cases} 1/2, & k = 16 \\ 1/2, & k = 24 \\ 0, & \text{otherwise.} \end{cases}$$

But what if $N = 41$? Then we cannot use coefficient matching since $M/41 \neq 2/5$ for any integer M.

But we can still find the 41-point DFT of $x[n]$ using `fft`, the result of which is shown in the figure. Now instead of a "clean line" we get peaks at 16 and 24 but the peaks are a bit "smeared out." Using a larger N, like $N = 401$ shown in the figure, tightens up the lines.



From the synthesis equation

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N} kn}$$

we see that the frequency associated with the kth DFT coefficient is

$$\boxed{\hat{\omega} = \frac{2\pi}{N} k.}$$

Combining this with the relationship (3d-3) we have $f_0/f_s = k/N$. However, this is only valid for $\hat{\omega} \leq \pi$. For $\hat{\omega} > \pi$ we must consider the equivalent frequency between $-\pi$ and 0. In summary:

$$f = \begin{cases} \frac{k}{N} f_s, & k = 0, \ldots, N/2 - 1 \\ \frac{N-k}{N} f_s, & k = N/2, \ldots, N-1. \end{cases} \tag{3d-4}$$

These are the formulas that any "digital spectrum analyzer" must use when displaying the DFT coefficients of samples of a continuous-time signal with the axes labeled in terms of Hz.

Further analysis

For concreteness, consider a periodic continuous-time signal $x(t)$ with period T, and consider sampling it with sampling interval $T_s \ll T$. Then the resulting discrete-time signal is

$$x[n] = x(nT_s)\,.$$

By the Fourier Series theorem, $x(t)$ may be expressed as a sum of complex exponential components:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\jmath 2\pi \frac{k}{T} t}$$

where

$$\alpha_k = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt.$$

Can we compute or approximate the α_k's from the samples of $x(t)$, *i.e.*, from $x[n]$? This question is answered by the following.

Fact F1

Let $x(t)$ be a periodic signal with period T, let $x[n] = x(nT_s)$ be the discrete-time signal formed by sampling $x(t)$ with sampling interval $Ts = T/N$, where $N \gg 1$, and let $X[k]$ denote the N-point DFT of $x[n]$. (It is easy to see that $x[n]$ is periodic with period N.) Claim:

$$\alpha_k \approx X[k] \text{ for } k \ll N.$$

Derivation:

To see how the α_k's can be approximated, we will use the fact that since $T_s \ll T$, $x(t)$ varies little over most T_s second intervals. Thus, it may be approximated with

$$x(t) \approx x(nT_s) = x[n]\,, \text{ for } nT_s \le t < nT_s + T_s.$$

We will also use the following approximation, the validity of which is discussed shortly:

$$e^{-\jmath 2\pi \frac{k}{T} t} \approx e^{-\jmath 2\pi \frac{k}{T} nT_s} = e^{-\jmath \frac{2\pi}{N} kn}, \qquad \text{when } nT_s \le t < nT_s + T_s.$$

With these approximations, we now proceed by rewriting the integral in the analysis formula as a sum of N integrals over intervals of length T_s seconds, where $N = T/T_s$:

$$\begin{aligned}
\alpha_k &= \frac{1}{T} \int_0^T x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt \\
&= \frac{1}{T} \sum_{n=0}^{N-1} \int_{nT_s}^{nT_s+T_s} x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt \quad \text{integrating over short intervals} \\
&\approx \frac{1}{T} \sum_{n=0}^{N-1} \int_{nT_s}^{nT_s+T_s} x[n]\, e^{-\jmath \frac{2\pi}{N} kn}\, dt \quad \text{using the above approximations} \\
&= \frac{1}{T} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} kn} \int_{nT_s}^{nT_s+T_s} dt \quad \text{rearranging terms} \\
&= \frac{1}{T} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} kn} T_s \quad \text{computing the integral} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} kn} \quad \text{using } N = T/T_s \\
&= X[k] \quad \text{the } k\text{th coefficient in } N\text{-point DFT of } x[n].
\end{aligned}$$

This fact shows that the kth Fourier coefficient α_k is approximately equal to the kth DFT coefficient $X[k]$ in an N-point DFT of $x[n]$. This means that the DFT coefficient $X[k]$ indicates the presence in $x(t)$ of the spectral component (*cf.* (3d-4)):

$$X[k]\, e^{\jmath 2\pi \frac{k}{T} t} \text{ at frequency } \frac{k}{T} = \frac{k}{NT_s} = \frac{k}{N} f_s,$$

where $f_\mathrm{s} = 1/T_\mathrm{s}$ is the sampling frequency. In other words, the component at frequency $\hat{\omega} = \frac{2\pi}{N}k$ in the discrete-time signal $x[n]$ represents a spectral component in the continuous-time signal $x(t)$ at frequency $\frac{k}{N}f_\mathrm{s}$.

On the other hand, the fact that $\alpha_k \approx X[k]$ seems to contradict the fact that there are infinitely many k's for a continuous-time periodic signal, but only finitely many $X[k]$'s. This apparent paradox is resolved by noting that the approximation

$$\mathrm{e}^{-\jmath 2\pi \frac{k}{T} t} \approx \mathrm{e}^{-\jmath \frac{2\pi}{N} kn}, \qquad \text{when } nT_\mathrm{s} \le t < nT_\mathrm{s} + T_\mathrm{s}$$

is valid when and only when the exponential varies little within each T_s second interval. Since the exponential is periodic with period T/k, this approximation is valid when and only when $T_\mathrm{s} \ll T/k$, or equivalently, when

$$k \ll \frac{T}{T_\mathrm{s}} = N.$$

Thus we see that the approximation $\alpha_k \approx X[k]$ is valid only when $k \ll N$. (Actually, it turns out to be fairly good as long as $k < N/2$.)

In summary, we have shown how to approximately compute the Fourier series coefficients $\{\alpha_k\}$ of a continuous-time signal from its samples. And the computation turns out to be the DFT!

We have also shown that the approximation is valid when $k \ll N = T/T_\mathrm{s}$. This indicates that where possible, one should choose the sampling interval T_s to be small enough so that the α_k's can be well approximated over whatever range of frequencies are of interest.

With this approximation for the Fourier series coefficients, one can now use the DFT coefficients to approximate the (two-sided) spectrum of the continuous-time signal $x(t)$ as follows

$$\left\{ \left(X^*[K], -\frac{K}{N}f_\mathrm{s}\right), \ldots, \left(X^*[1], -\frac{1}{N}f_\mathrm{s}\right), (X[0], 0), \left(X[1], \frac{1}{N}f_\mathrm{s}\right), \ldots, \left(X[K], \frac{K}{N}f_\mathrm{s}\right), \right\},$$

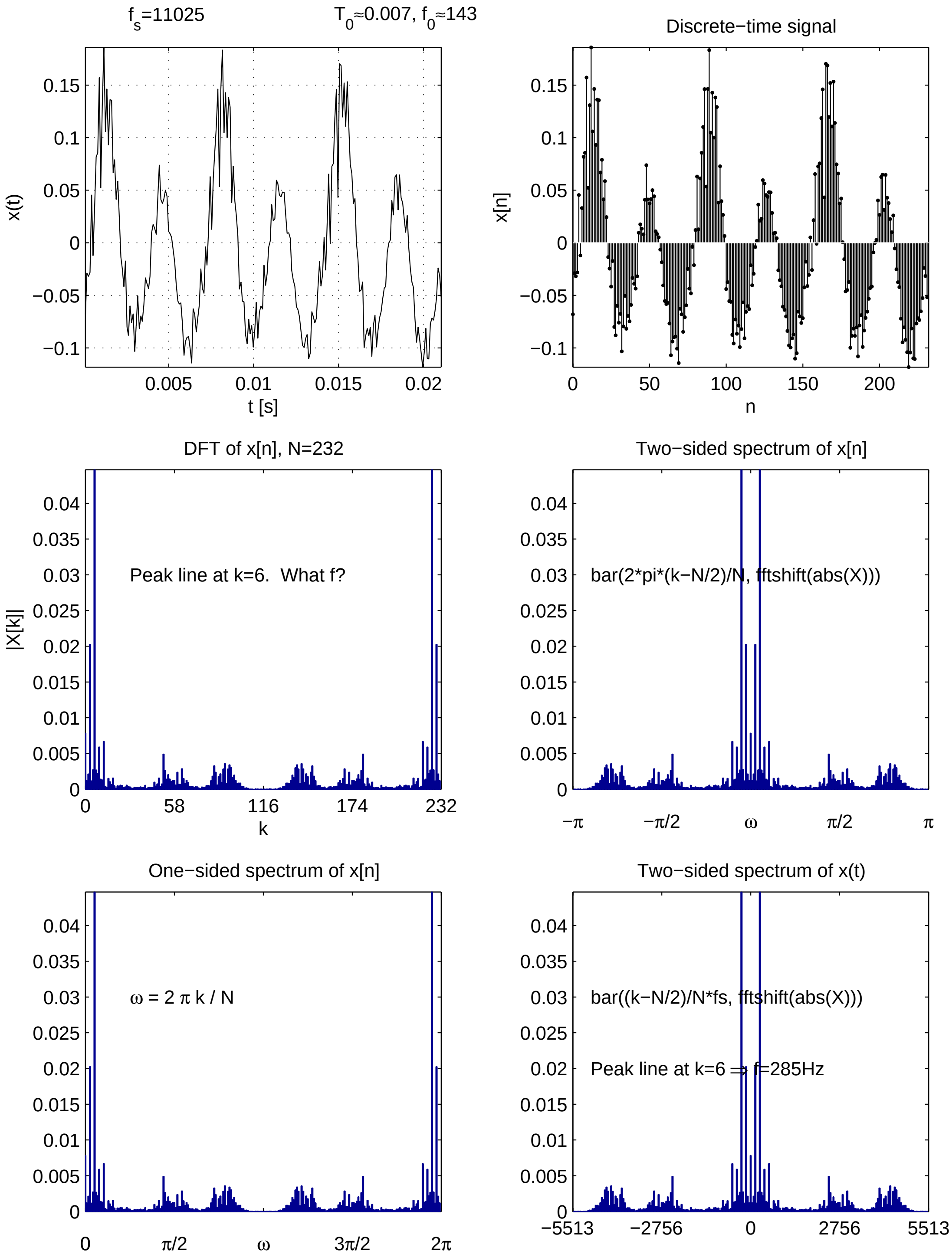
where K is the largest value of k for which we believe $\alpha_k \approx X[k]$. Note that we are, in effect, approximating the spectrum as having no components above frequency $K f_\mathrm{s}$.

Although we know the approximation is valid only for $k \ll N$, it is quite common to use the entire set of DFT coefficients in an approximation for the spectrum of $x(t)$. Specifically, it is common to plot the one-sided spectrum

$$\left\{ (X[0], 0), \left(X[1], \frac{1}{N}f_\mathrm{s}\right), \left(X[2], \frac{2}{N}f_\mathrm{s}\right), \ldots, \left(X[N-1], \frac{N-1}{N}f_\mathrm{s}\right), \right\}.$$

However, when interpreting such a plot, one must recall that such a spectrum is accurate only for values of $k \ll N$. Moreover, when $k > N/2$, the term $X[k] = X^*[N-k]$. Thus if anything, for $k > N/2$, the spectral line shown at frequency $\frac{k}{N}f_\mathrm{s}$ is indicative of what happens at frequency $\frac{(N-k)}{N}f_\mathrm{s}$, *not* at what happens at frequency $\frac{k}{N}f_\mathrm{s}$.

Example. The following figure shows a continuous-time signal, its samples, the magnitudes of its DFT coefficients, the one-sided magnitude spectrum of the discrete-time signal, the two-sided magnitude spectrum of the discrete-time signal, and the approximate continuous-time spectrum. The sampling rate is $f_s = 11,025$ samples/sec.



G. Bandwidth

One of the primary motivations for assessing the spectrum of signal is to find the range of frequencies occupied by it. This range is often called the signal's "band of spectral occupancy," or, more simply, its **frequency band**. The width of the frequency band is called the **bandwidth**.

As one example, signals with non-overlapping spectra do not interfere with each other. So if we know the frequency band occupied by each of a set of signals, we can determine if they interfere. As another example, certain communication media, *e.g.*, a wire, limit propagation to signals with spectral components in a certain range. If we know the frequency band occupied by a signal, we can determine if it will propagate.

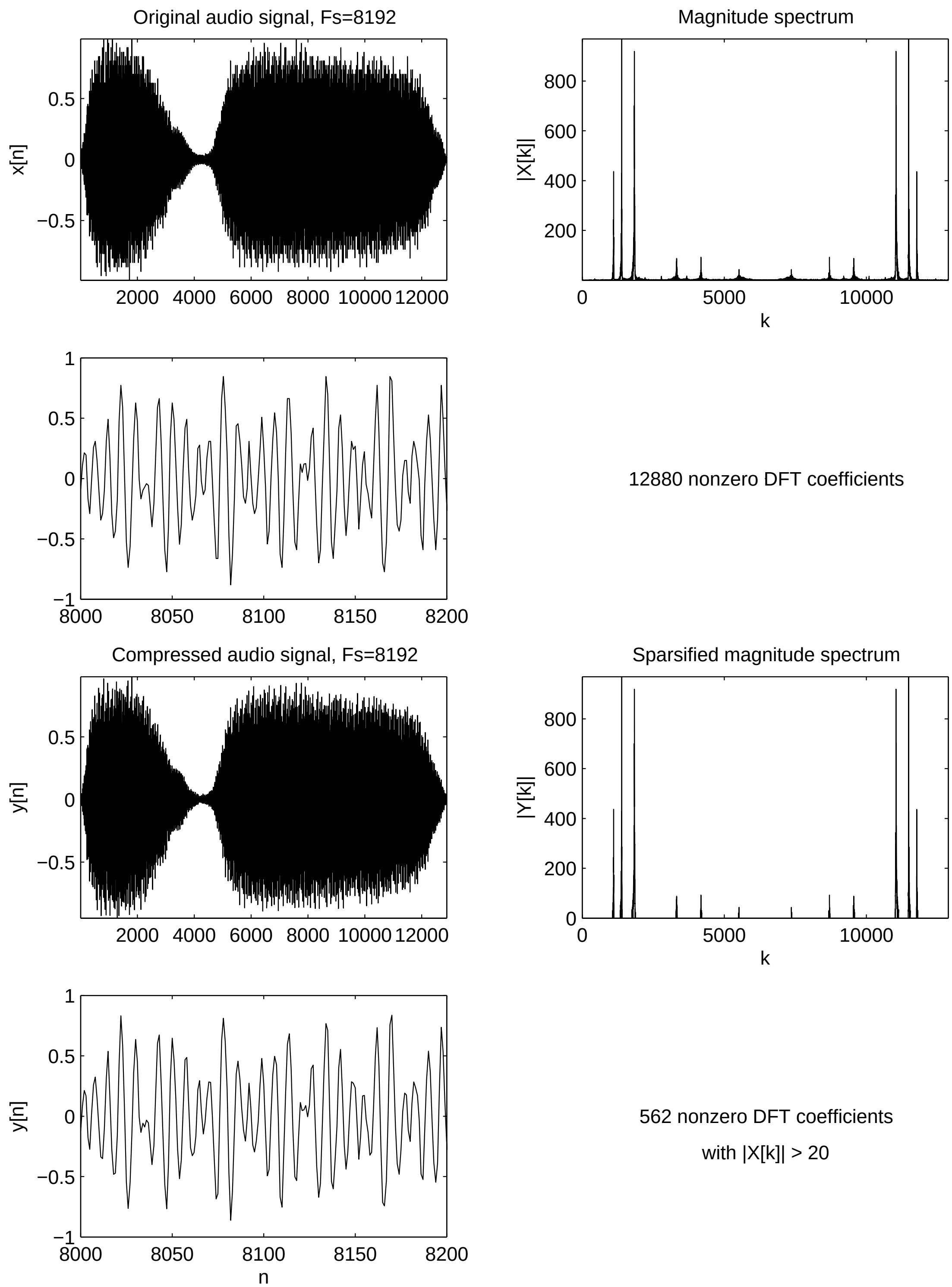
Most signals of practical interest, such as that shown in the previous section, have spectral components extending over a broad range of frequencies. We are not really interested in the entire range of frequencies over which the spectrum is not zero. Rather we are interested in the range of frequencies over which the spectrum is "significantly large." As such, we need a definition of "significantly large" to define the concepts of "frequency band" and "bandwidth." There are a number of such definitions in use. The definition given below is based on one such definition.

Definition:

The "band of spectral occupancy" or **frequency band** of a signal $x[n]$ is the smallest interval of frequencies that includes all frequencies at which the magnitude spectrum is at least one half as large as the maximum value of the magnitude spectrum.

Signal compression

The following figure, discussed in detail in lecture, describes how discarding small frequency components can greatly reduce storage (like MP3) with only modest signal distortion.



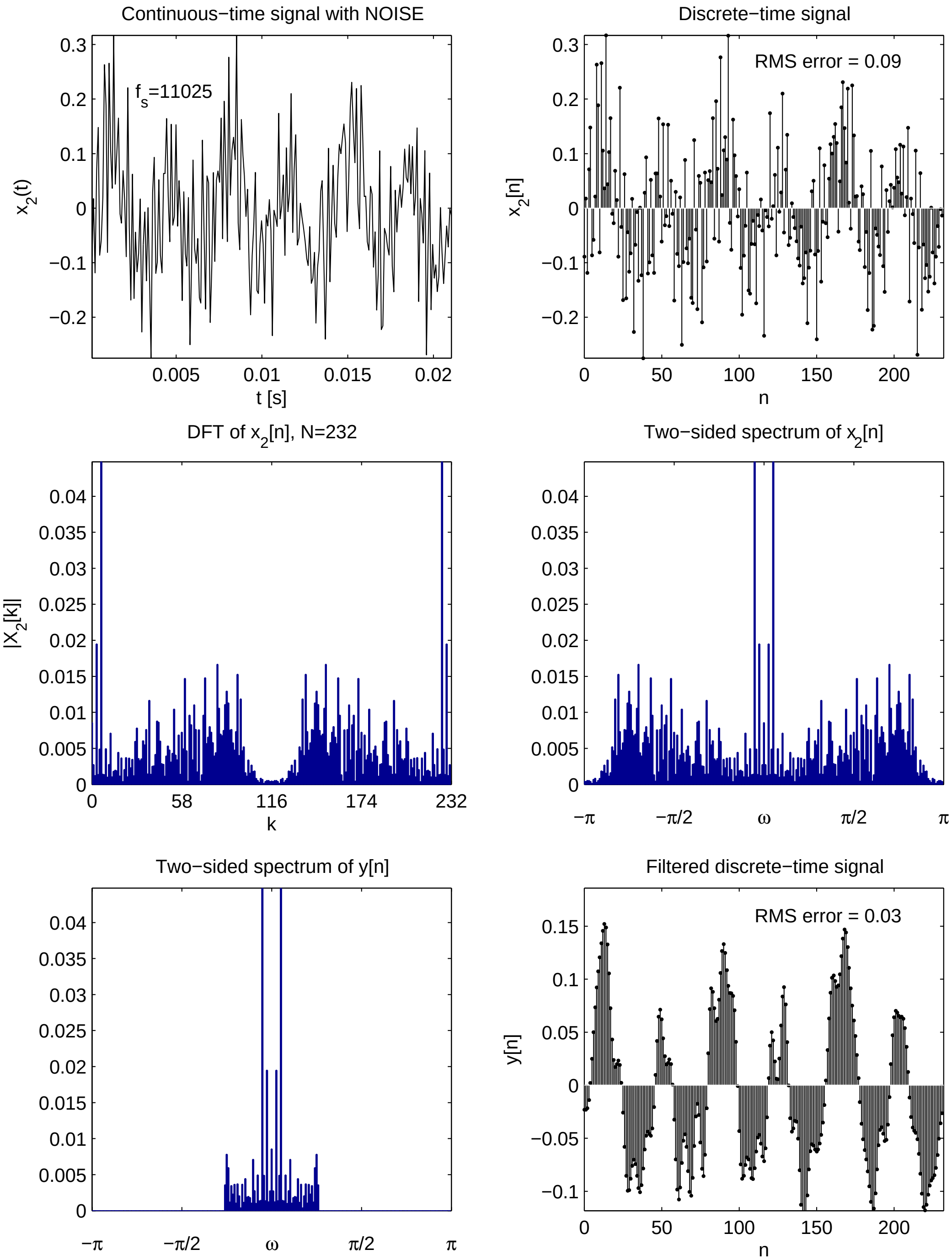
Noise removal by filtering with the DFT

Here are three (of many) applications of the DFT.

- Signal compression (*e.g.*, MP3, JPEG)
- Digital spectrum analyzers (*e.g.*, digital 'scopes)
- Noise removal (filtering)

We have discussed the 1st two applications, now we turn to the third.

The following figure, discussed in detail in lecture, shows a signal with noise, and shows that removing the high frequency components can eliminate a lot of noise yet retain the dominant signal components.



The family of Fourier transforms

In introducing the concept of **spectra**, we mentioned that there are various definitions of spectra depending on the type of signal being considered.

Periodic signals

Continuous-time T-periodic
Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\jmath 2\pi \frac{k}{T} t}$$

Analysis (**Fourier series**)

$$\alpha_k = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt$$

Frequencies:

$$f \in \left\{ \frac{k}{T} : k \in \mathbb{Z} \right\}.$$

Discrete-time N-periodic
Synthesis

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{\jmath \frac{2\pi}{N} kn}$$

Analysis (**Discrete Fourier transform**)

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-\jmath \frac{2\pi}{N} kn}$$

Frequencies:

$$\hat{\omega} \in \left\{ \frac{2\pi}{N} k : k = 0, \ldots, N-1 \right\}.$$

Aperiodic signals

Continuous-time
Synthesis

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{\jmath 2\pi f t}\, df$$

Analysis (**Fourier transform**)

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-\jmath 2\pi f t}\, dt$$

Frequencies:

$$f \in \mathbb{R}.$$

Discrete-time
Synthesis

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\hat{\omega}) e^{\jmath \hat{\omega} n}\, d\hat{\omega}$$

Analysis (**Discrete-time Fourier transform**)

$$X(\hat{\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-\jmath \hat{\omega} n}$$

Frequencies:

$$\hat{\omega} \in [-\pi, \pi].$$