First Problem Assignment

EECS 401

Due on January 12, 2007

PROBLEM 1 (15 points) Fully explain your answers to the following questions.

(a) If events A and B are mutually exclusive and collectively exhaustive, are A^c and B^c mutually exclusive?

> **Solution** $A^c \cap B^c = (A \cup B)^c = \Omega^c = \varnothing$. Thus the events A^c and B^c are mutually exclusive.

(b) If events A and B are mutually exclusive but not collectively exhaustive, are A^c and B^c collectively exhaustive?

> **Solution** Let $C = (A^c \cup B^c)^c$, that is the part that is not contained in $A^c \cup B^c$. Using De Morgan's Law $C = A \cap B = \varnothing$. Thus, there is nothing that is not a part of A^c or B^c. Hence, A^c and B^c are collectively exhaustive.

(c) If events A and B are collectively exhaustive but not mutually exclusive, are A^c and B^c collectively exhaustive?

> **Solution** As in previous part, let $C = (A^c \cup B^c)^c = A \cap B$ which is not null. Thus, A^c and B^c are not collectively exhaustive.

PROBLEM 2 (10 points) Joe is a fool with probability 0.6, a thief with probability 0.7, and neither with probability 0.25.

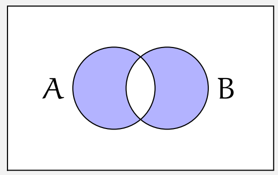
(a) Determine the probability that he is a fool or a thief but not both.

> **Solution** Let A be the event that Joe is a fool and B be the event that Joe is a thief. We are given that
>
> $$P(A) = 0.6$$
> $$P(B) = 0.7$$
> $$P((A \cup B)^c) = 0.25$$
>
> This implies:

$$P\big(A \cup B\big) = 1 - P\big((A \cup B)^c\big) = 0.75$$
$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.55$$

The event that he is a fool or a thief but not both is given by $(A \cap B^c) \cup (A^c \cap B)$. Looking at the Venn diagram, the probability should be:

$$P\big((A \cap B^c) \cup (A^c \cap B)\big) = P(A) + P(B) - 2P(A \cap B) = 0.2 \tag{1}$$



We can also derive this as follows

$$(A \cap B^c) \cup (A^c \cap B) = (A \cup B) \cap (A \cap B)^c$$

Thus,

$$\begin{aligned} P\big((A \cup B) \cap (A \cap B)^c\big) &= P(A \cup B) + P\big((A \cap B)^c\big) - P\big((A \cup B) \cup (A \cap B)^c\big) \\ &= P(A \cup B) + 1 - P(A \cap B) - P(S) = \boxed{P(A \cup B) - P(A \cap B) = 0.2} \end{aligned}$$

This is the same expression as in (1).

(b) Determine the conditional probability that he is a thief, given that he is not a fool.

Solution We need to find $P(B|A^c)$. We know that

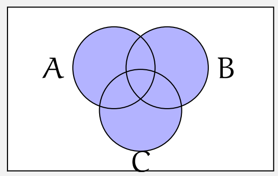
$$P(B|A^c) = \frac{P(B \cap A^c)}{P(A^c)} = \frac{P(B) - P(B \cap A)}{1 - P(A)}$$
$$= \frac{0.7 - 0.55}{1 - 0.6} = \boxed{0.375}$$

PROBLEM 3 (35 points) Express each of the following events in terms of the events A, B and C as well as the operations of complementation, union and intersection:

(a) at least one of the events A, B , C occurs;

Solution

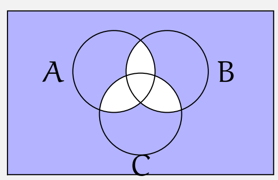
$A \cup B \cup C$



(b) at most one of the events A, B , C occurs;

Solution

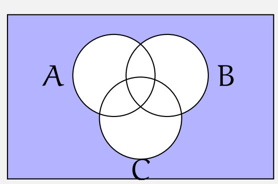
$[(A \cap B) \cup (B \cap C) \cup (C \cap A)]^c$



(c) none of the events A, B , C occurs;

Solution

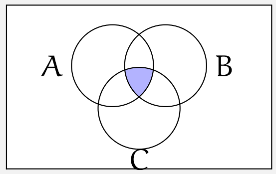
$A^c \cap B^c \cap C^c$



(d) all three events A, B, C occur;

Solution

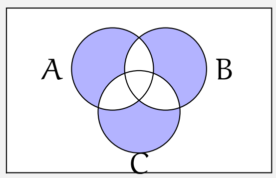
$A \cap B \cap C$



(e) exactly one of the events A, B, C occurs;

Solution

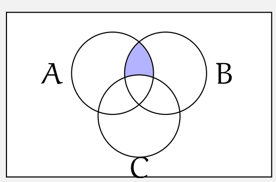
$[A \cap (B^c \cap C^c)] \cup [B \cap (C^c \cap A^c)] \cup [C \cap (A^c \cap B^c)]$



(f) events A and B occur, but not C;

Solution

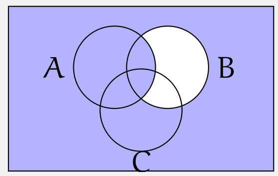
$A \cap B \cap C^c$



(g) either event A occurs or, if not, then B also does not occur.

Solution

$A \cup (A^c \cap B^c)$



In each case draw the corresponding Venn diagram.

PROBLEM 4 (14 points) Suppose A and B are two events with known probabilities.

(a) Can you compute $P(A \cup B)$ in terms of $P(A)$ and $P(B)$? If so, explain. If not, find the largest and smallest possible values $P(A \cup B)$ can take in terms of $P(A)$ and $P(B)$ and give examples in which these values can be attained (i.e., give upper and lower bounds for $P(A \cup B)$.).

Solution No, $P(A \cup B)$ can not be computed in terms of $P(A)$ and $P(B)$ only. We know that,

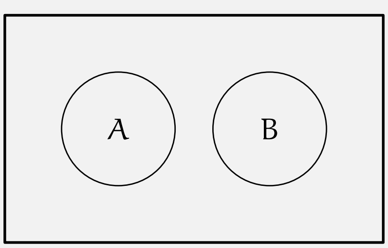
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Knowing only $P(A)$ and $P(B)$ does not give complete information about $P(A \cap B)$. However we know that

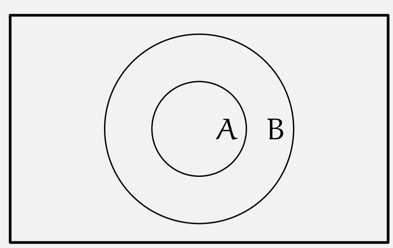
$$0 \leqslant P(A \cap B) \quad \text{and} \quad P(A \cap B) \leqslant \min\{P(A), P(B)\}$$

$$\therefore \quad \boxed{\max\{P(A), P(B)\} \leqslant P(A \cup B) \leqslant P(A) + P(B)}$$

The following cases can achieve the lower and the upper bound



Upper bound on Union



Lower bound on Union

(b) Repeat previous part with $P(A \cap B)$ instead of $P(A \cup B)$.

Solution As shown in the previous part

$$0 \leqslant P(A \cap B) \leqslant \min\left\{P(A), P(B)\right\}$$

Also, note the case of part (a) that achieves the upper bound on $P(A \cup B)$ also achieves the lower bound on $P(A \cap B)$. Similarly, the case of part (a) that achieves the lower bound on $P(A \cup B)$ also achieves the upper bound on $P(A \cap B)$.

PROBLEM 5 (14 points) Prove that $A \subset C$ if $A \subset B$ and $B \subset C$. (*In order to prove this you need to argue that for every element $x \in A$ it is true that $x \in C$ and that there is at least one element $y \in C$ such that $y \notin A$.*)

Solution

$$A \subset B \text{ means that } \forall x \in A \implies x \in B \text{ and } \exists y \in B \text{ s.t. } y \notin A \quad (1)$$
$$B \subset C \implies \forall x \in B \implies x \in C \text{ and } \exists y \in C \text{ s.t. } y \notin B \quad (2)$$

From (1) and (2) we get that

$$\begin{aligned} & x \in A \text{ means that } x \in B \text{ and } \forall x \in B, x \in C \\ \text{Thus,} \quad & \forall x \in A \implies x \in C \\ \text{i.e.,} \quad & A \subseteq C \end{aligned} \quad (3)$$

Furthermore from (2) we get that

$$\begin{aligned} & \exists y \in C \text{ s.t. } y \notin B \\ \text{(1) implies} \quad & y \notin B \implies y \notin A \\ \text{i.e,} \quad & \exists y \in C \text{ s.t. } y \notin A \end{aligned}$$

Combining this with (3) we get that $A \subset C$.

PROBLEM 6 (12 points) Let A and B be two sets.

(a) Show that $(A^c \cap B^c)^c = A \cup B$ and $(A^c \cup B^c)^c = A \cap B$

Solution

$$\begin{aligned}(A^c \cap B^c)^c &= (A^c)^c \cup (B^c)^c && \text{by De Morgan's Law} \\ &= A \cup B && because\ (A^c)^c = A\end{aligned}$$

Similarly:

$$\begin{aligned}(A^c \cup B^c)^c &= (A^c)^c \cap (B^c)^c && \text{by De Morgan's Law} \\ &= A \cap B && because\ (A^c)^c = A\end{aligned}$$

(b) Consider rolling a six-sided die once. Let A be the set of outcomes where an odd number comes up. Let B be the set of outcomes where a 1 or a 2 comes up. Calculate the sets on both sides of the equalities in part 1, and verify that the equalities hold.

Solution $A = \{1, 3, 5\}$ and $B = \{1, 2\}$. Then $A^c = \{2, 4, 6\}$ and $B^c = \{3, 4, 5, 6\}$.

$A^c \cap B^c = \{4, 6\}$, thus $(A^c \cap B^c)^c = \{1, 2, 3, 5\}$.

$A \cup B = \{1, 2, 3, 5\}$. Thus $(A^c \cap B^c)^c = A \cup B$.

Similarly, we can show that $(A^c \cup B^c)^c = A \cap B = \{1\}$.