Multi-Agent Collaborative Decision Making: Constrained Event Algebras, New Logics and Their Game Theoretic Semantics

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Happy Birthday Demos!

Νά ζήσεις Δήμος και χρόνια πολλά …
May you live many happy years Demos …

παντού νά σκορπίζεις τής γνώσης το φώς !
may you spread everywhere the light of knowledge

from a popular Greek birthday song
Non-commutative Probability

Examples

• Directed Information and Network Information Theory

• In many types of networks ~ e.g. Heterogeneous sensor networks
  – Heterogeneous interdependent data
  – Collected by different sensors
  – Stored in different ways
  – Collaboration ~ sensor fusion
Non-commutative Probability

Examples

• Humans are final decision makers ~ an indispensable part of these networks
  – Collaborative inference?
  – Collaborative cognition?
  – Collaborative decision making?

• How to model humans? They are not causal!
Non-commutative Probability
Examples

• Multi-agent control and decision making under various “information patterns”
  – Who knows what and when?
  – How should common and non-common information be treated/combined?
  – How are stochastic models constructed?
  – Are the $\sigma$ - algebras too rich – thus constraining solutions to be impractical?

• What are the proper stochastic models?
Fundamental Problem

- **Effective and efficient analysis, fusion and exploitation** of these vast amounts of heterogeneous data and their transformation to *meaningful, valuable and actionable information*

- **Resource constraints**: communication, memory, real-time or finite time

- Interaction between dynamic semantic knowledge and spatiotemporal statistical characteristics

- **Need deeper understanding of the relationships between probabilistic models/methods and logic models/methods**
Multiple Interacting Dynamic Multigraphs

- Multiple Interacting Multigraphs
  - **Nodes**: agents, individuals, groups, organizations
  - Directed graphs
  - **Links**: ties, relationships
  - Weights on links: value (strength, significance) of tie
  - Weights on nodes: importance of node (agent)
- **Value directed graphs with weighted nodes**
- **Real-life problems**: Dynamic, time varying graphs, relations, weights
Back to Basics

• Revisit events and event algebras
• Revisit logic of events and its implications
• Understand constrained event algebras
• Systematic construction of models from observations
• Revisit the meaning of state
• Revisit the meaning of networks
Constrained Event Algebras

- **Simple example** (Willems): Noisy resistor
  - Deterministic behavior
    \[ B = \left\{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V = RI \right\} \]
  - Stochastic system
    \[ E = \left\{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V - RI \in A \text{ with } A \subseteq \mathbb{R} \text{ Borel} \right\} \]
    \[ P(E) = \frac{1}{\sqrt{2\pi\sigma}} \int_{A} e^{-\frac{x^2}{2\sigma^2}} dx. \]
  - The vector \( \begin{bmatrix} V \\ I \end{bmatrix} \) is not a classical random vector
  - Only cylinders with rays parallel to \( V=RI \) are events that are assigned probabilities

- Channels, interfaces, interconnected systems, complementary \( \sigma \)-algebras, open vs closed systems
Directed Information
(Marko, Massey, Kramer …)

• Correlation does not imply causation

• *Causal relationships correspond to stable deterministic mechanisms*, by which one set of variables (the causes), together with some possibly unobserved exogenous disturbances, may affect another set of variables (the effects).

• Inferring causal relationships requires active experimentation that intervenes into some of these mechanisms

• Quite different from estimating statistical effects

• “*statistical dependence, unlike causality, has no “inherent directivity.”*” (Massey)
Directed Mutual Information

Simple example:

• Two binary valued r.v. $X$, $Y$
• Decision maker observes and attribute of $X$ before observing an attribute of $Y$ and uses
  \[ P_{X|Y}(0|1) = P_{X|Y}(1|0) = \varepsilon. \]
  estimate $Y$ based on $X$
• Next $Y$ is observed before $X$, followed by estimation of $X$ based on
  \[ P_{Y|X}(0|1) = P_{Y|X}(1|0) = \delta. \]
• If $\varepsilon \neq \delta$, there is no joint distribution $P_{X,Y}$ of the attributes $X$ and $Y$ that is consistent with these conditional distributions -- underlying system of equations will overdetermine the marginals of $X$, $Y$
• Mutual Information cannot be defined; not useful
Directed Mutual Information

Directed Information from $X^n$ to $Y^n$:

$$I(X^n \to Y^n) = \sum_{i=1}^{n} I(X^i; Y_i | Y^{i-1}) = E \left[ \log \frac{P(Y^n || X^n)}{P(Y^n)} \right]$$

$$P(Y^n || X^n) = \prod_{i=1}^{n} P(Y_i | X^i, Y^{i-1})$$

- **Mutual Information:**
  $$I(X^n; Y^n) = E \left[ \log \frac{P(Y^n | X^n)}{P(Y^n)} \right]$$
  $$P(Y^n | X^n) = \prod_{i=1}^{n} P(Y_i | X^n, Y^{i-1})$$

- Mutual Information symmetric, permutation invariant
- Dir. Information asymmetric $I(X^n \to Y^n) \neq I(Y^n \to X^n)$
- Can be extended to complex examples; dynamical systems with multiple feedbacks -- interventions
- **Networked systems version? Information flows?**
Non-commutative Probability

Examples

• Tracking and identification of moving objects using multiple-cameras

• High-level activity detection and anomalous activity mining from multiple perspectives

• Trust in social networks
  – Neuropsychological studies, emotional activation, interpersonal relationships, trust, decisions relying on trust.

• Human judgments and noncommutativity
  – Based on indefinite state, create than record, disturb each other, do not obey classic logic, law of total probability does not hold
Non-commutative Probability Examples

- Intimately related with need to have probability models and event logic models that *allow by incompatible events*, and *questions*
- Cause measurements and observations to be non-commutative
- Incompatible questions cannot be evaluated on the same basis, so that they require setting up separate sample spaces
- These new probability models provide more flexibility for assigning probabilities to events, and they do not require forming all possible joint probabilities, which is needed to provide the foundations for the inference, cognition and decision making problems of interest.
Non-commutative Probability Examples

• Human cognition and decision making
  – Disjunction effect
  – interference between categorization and decision making
  – Conjunction fallacy
  – Compositionality in the semantics of cognitive information processing
  – Related to question order effects that cannot be explained by classical probability models

• Classical Kolmogorov-like models cannot explain any of the observed phenomena and measurements, in examples.

• Recent studies utilizing the alternative quantum-like probabilities and logics have shown considerable agreement with the experimental data in these phenomena.
Non-Commutative Probability Models – New Logics

• **Key idea**: interaction between measurements by different agents and between system states and dynamics and measurements (Baras, 1979)
  – Now investigated vigorously in information retrieval systems (van Rijsbergen, 2004)
  – Asynchrony and concurrency

• **Key challenge**: understand the fundamentals of information collection and information flow in multi-agent stochastic control systems

• Witsenhausen’s model of information patterns and its limitations

• Need for new non-commutative probability models – **new logics** -- projections in Hilbert space
The Setting and the Problems

• $N$ agents, local states, local times
• Measurements and hypotheses supported and interpreted by local states
• **Static problems**: distributed detection and estimation problems
• **Simple dynamics**: like in information retrieval systems
• **Complex system dynamics**: full interactions between measurements and measurements and controls
• Must unify the probabilistic and logical aspects in a consistent way
Outline

• New Probability/Logic Models in Information Retrieval

• New Probability Models in Human Inference and Cognition

• New Probability/Information Models in Distributed Inference and Control
Information Retrieval (IR): Why a fresh look?

Van Rijsbergen (2004)

- Need for a single universal framework
- Need for formal (analytic) performance evaluation; e.g. relevance (relevance requires information theories that include content)
- Need for formal specification and representation of IR processes to reason and design new algorithms
- Need IR models and algorithms that provably apply to all media objects: text, image, audio, etc.
Relevance in IR

• Relevance is subjective – varies from user to user
• Relevance depends on the state of the user – but it changes as user acquires information
• Plenty of evidence that relevance of a document to a user changes as the user interacts with the system
• Probabilistic assessment of relevance – but on what logic?
• “Aboutness” of an object is not inherent but results from queries – i.e. via interactions from the system
Interaction Protocol for IR

- $x$ the document, $t$ the term
- $A \sim \text{“Aboutness”} – \text{is } x \text{ about } t$ ?
- $R \sim \text{“Relevance”} – \text{is } x \text{ relevant }$ ?
- $A$ and $R$ interact – second time of applying $A$ answer may not be the same
- Need for interaction protocol between users and information spaces – interaction logic – Latent Semantic Indexing in IR
Elementary IR

- Objects ~ documents; Can decide if for each object a particular attribute applies to it
- e.g. text about politics; image about churches
- IR deciding – indexing
- Objects \( \{x, y, z, \ldots\} = \Omega \) and predicates \( \{P, Q, \ldots\} \)
- \( |P| = \{x \mid P(x) \text{ is true}\} \subset \Omega \)
- Composite queries
  \[
  |P \land Q| = \{x \mid P(x) \text{ is true and } Q(x) \text{ is true}\} \subset \Omega
  \]
  \[
  |P \lor Q| = \{x \mid P(x) \text{ is true or } Q(x) \text{ is true}\} \subset \Omega
  \]
  \[
  |\neg Q| = \{x \mid \text{it is not true that } Q(x) \text{ is true}\} \subset \Omega
  \]
- Formal language -- set-theoretic operations
**Boolean IR**

- \([|P \land Q|] = [|P|] \cap [|Q|]\)
- \([|P \lor Q|] = [|P|] \cup [|Q|]\)
- \([|\neg P|] = \Omega - [|P|]\)

Or more complex expressions:
- \([| (P \land Q) \lor \neg R |] = ( [|P|] \cap [|Q|] ) \land [|\neg R|] \)

**Principle of Compositionality**
- Do not need to know anything about the structure of \(\Omega\)

**Ample experimental evidence**: IR based on this simple model does not deliver the required performance
Performance of IR systems

- **Precision**: ability to retrieve the relevant objects
- **Recall**: Retrieve as few as possible of the non-relevant objects
- Need to extend the structure of $\Omega$ -- add the counting measure
- $A = \text{subset of relevant documents}$
- $B = \text{subset of retrieved documents}$
- **Precision** = $|A \cap B| / |B|$
- **Recall** = $|A \cap B| / |A|$
- **Composite Effectiveness measure – E-measure**

$$E = |A \triangle B| / (|A| + |B|) = (|A \cup B| - |A \cap B|) / (|A| + |B|)$$
Problems with this IR Model

• Can extend to a more general probabilistic measure on $\Omega$, but ...

• Predicate $B = \{\text{retrieved}\}$ – not known in advance – usually specified algorithmically

• Predicate $A = \{\text{relevant}\}$ – is user dependent

• Not known in advance and non static

• But predicates interact – aboutness predicates interact with relevance predicates – cognitive activity has occurred that changed the “state” of the observer from one measurement moment to the next (Borland, 2000)
Compatibility and Logic Type

• **Compatibility** between predicates

\[ X = ( X \land Q) \lor ( X \lor \neg Q) \]

• Does not hold when predicates are **incompatible**

• Distribution law of logic – holds in the Boolean model:

\[ X = ( X \land Q) \lor ( X \lor \neg Q) = X \land ( Q \lor \neg Q) \]

• Boolean Model too strong if we want to model incompatibility

• Inference – Deduction Theorem

\[ A \land B \models C \iff A \models B \rightarrow C \]
Inverted Files – Non Boolean Logics

- Objects, attributes, “buckets”
- \( \Omega \) the objects, \( V \) the possible attributes
  
  \[ \text{tr}(A) = \{ t \in V \mid \alpha \Delta t \text{ for all } \alpha \in A \} \text{ (indexing)} \]
  
  \[ \text{in}(T) = \{ \alpha \in \Omega \mid \alpha \Delta t \text{ for all } t \in T \} \text{ (get objects)} \]
  
  \( \Delta \) a relation in \( \Omega \times V \)

- Naïve set theory \( H \cup L = \text{in}(\text{tr}(H \cup L)) \)

- But in IR : \( H \cup L \subset \text{in}(\text{tr}(H \cup L)) \) – Non-Boolean logic

- ( \( \text{tr}, \text{in} \) ) is actually a \textbf{Galois connection} between the partially ordered sets \( \wp(\Omega) \) and \( \wp(V) \)

- Examples where distributive law fails – especially in the context of images and image features

- \textbf{Duality}: logic of object classes – logic of attribute classes
Conditional Logic in IR

• Document: a set of assertions or propositions
• Query: a single assertion or proposition
• Document relevant to a query if it implies the query
• Typically a query is not implied by any document – failure of Boolean model

• Projection operators $E$, on a Hilbert space $H$
• Linear, self-adjoint, idempotent  -- $E = E^* = E^2$
  An orthomodular lattice – $E \leq F$ iff $FE = E$
• To each $E$ ~ a subspace $|E| = \{ x \mid Ex = x, x \in H \}$
• $E^\perp = I - E$, $E \land F = EF$, $E \lor F = E + F - EF$
  Complement, meet, joint
Lattice of Projections

- Conditional $E \rightarrow F$
  
  $[|E \rightarrow F |] = \{x \mid FEx = Ex, x \in H\} =
  
  = \{x \mid F^\perp Ex = 0, x \in H\}$

- A subspace – has the character of implication – but the lattice of subspaces is not distributive – Non Boolean Logic

- $E \rightarrow F$ is a conditional, that satisfies only weak forms of transitivity, weakening and contraposition

- $E \rightarrow F = E^\perp \lor (E \land F)$

- Can define compatibility relation $K$ now
  
  $A K B$ iff $A = (A \land B) \lor (A \land B^\perp)$
Lattice of Projections cont.

- \( R \sim \) projector on subspace of relevant objects
- \( E \sim \) projector onto the subspace of objects about observable \( E \)
- Then compatibility \( \sim R = \left( R \land E \right) \lor \left( R \land E\perp \right) \)
- The non-classical IR view: an object may be about some topic or its negation once observed, but before observation it may be neither
- IR – Stalnaker conditional – \( S_A(x) \) the nearest world to \( x \) in which \( A \) is satisfied – \( S_A(x) = Ax \) – Projection Theorem!
The New Logic of IR

• Any object/document ~ a normalized vector \( x \) in a finite dimensional Hilbert space (possibly complex) – the corresponding ray

• Similarly a query \( y \)

• Similarity matching – compute \( (x, y) \)

• **Gleason’s theorem**: measure \( \mu \) on subspaces of a separable Hilbert space \( H \) \( \exists \) a positive, self-adjoint operator \( T \) of trace class s.t. \( \mu(L) = \text{tr}(TP_L) \) – probability measures \( \text{tr}(T) = 1 \) – density operators \( \rho \) – generalized queries

• In IR similarity to sets of vectors, or clusters – coordination level matching, relevance feedback
Many Applications

• Dynamic clustering – feedback – relevance – repeated queries ~ simply adjust $\rho$

• User has seen a number of objects and decided about their relevance, how to guide her to a number of objects likely to be relevant

• Incorporate new relevance assessments and calculate probability of objects to be considered next

• Language modeling

• Probability of conditionals (contexts) and Information Theory
Outline

• New Probability/Logic Models in Information Retrieval

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Bounded Rationality (H. Simon)

- Leibniz dreamed to reduce rational thinking to one universal logical language: the *characteristica universalis*.
- Rational decisions by humans and animals in the real world are bound by limited time, knowledge, and cognitive capacities. These dimensions are lacking classical models of logic and decision making.
- Some people such as Gigerenzer see Leibniz' vision as a unrealistic dream that has to be replaced by a toolbox full of heuristic devices (lacking the beauty of Leibniz' ideas)
Paradoxes/Puzzles of Bounded Rationality

- Order effects: In sequences of questions or propositions the order matters: \((A ; B) \neq (B ; A)\) (see survey research)
- Disjunction fallacy: Illustrating that Savage's sure-thing principle can be violated
- Graded membership in Categorization: The degree of membership of complex concepts such as in "a tent is building & dwelling" does not follow classical rules (Kolmogorov probabilities)
- Others: Conjunction puzzle (Linda-example), Ellsberg paradox, Allais paradox, prisoner dilemma, framing, ¼
Natural Concepts Do NOT Form Boolean Algebras

- **P. Gärdenfors**: *The Geometry of Thought* (2000)
  Concepts as convex spaces
- The intersection of convex sets is convex again, but union and complement are not.
- **Hans Primas**: Convex sets form orthomodular lattices.
- **Orthomodularity**: If $A \leq B$ then $A = (A \cap B) + (A \cap B^\perp)$
Possible Worlds and Vectors

- **Possible worlds**: Isolated entities which are used for modeling propositions (sets of possible worlds)

- **Vector states**: abstract objects which form vector spaces.

  - The *addition* of two vectors is an operation which describes the superposition of possibly conflicting states
  
  - The *scalar product* is an operation which describes the similarity of two states
  
  - *Projections* are operators that map vector spaces onto certain subspaces
## Comparison

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Puzzle: Probability Judgements
Violate Kolmogorov’s Axioms

- A Boolean algebra over \( W \) is a set \( \mathcal{F} \) of subsets of \( W \) [events, possibilities] that contains \( W \) and is closed under union and complementation (intersection).

- Normalized additive measure function
  - \( P(A \cup B) = P(A) + P(B) \) for disjoint sets \( A \) and \( B \)
  - \( P(W) = 1 \)

- Consequences:
  - \( P(A \cap B) \leq P(A) \), \( P(A \cup B) \geq P(A) \) [monotonicity]
  - \( P(A) + P(B) - P(A \cap B) \leq 1 \) [additivity I]
  - \( P(A) + P(B) - P(A \cup B) \leq 1 \) [additivity II]
Tversky and Shafir (1992) show that significantly more students report they would purchase a nonrefundable Hawaiian vacation if they were to know that they have passed or failed an important exam than report they would purchase if they were not to know the outcome of the exam.

\[
P(A|C) = 0.54 \quad P(A|\neg C) = 0.57 \quad P(A) = 0.32
\]

\[
P(A) = P(A|C)P(C) + P(A|\neg C)P(\neg C)
\]

since \( CA \cup (\neg C)A = A \) (distributivity)

The ‘sure thing principle’ is violated empirically!
Puzzle: Complementarity

• Two Boolean descriptions are said to be complementary if they cannot be embedded into a single Boolean description.

• Unicity: In classical probability theory, a single sample space is proposed which provides a complete and exhaustive description of all events that can happen in an experiment.
  – If unicity is valid, then complementarity does not exist
  – If complementarity exists then unicity cannot be valid

• Examples: physical time/mental time; physical/mental objectivity (mind/body); Jung’s rational/irrational functions.
All cartographic maps are valid only *in the small*, i.e. locally.

Hans Primas
Puzzle: Question Order Effects for Attitude Questions

Puzzle: Asymmetric Similarities

• Korea is similar to China \(\text{vs.} \) China is similar to Korea
• Chicago's linebackers are like tigers \(\text{vs.} \) *Tigers are like Chicago's linebackers
• From a classical perspective this is puzzling:
  \[ \text{sim} (X, Y) = f(\text{distance} (X, Y)) \]
  \[ \text{sim} (Y, X) = f(\text{distance} (Y, X)) \]
  But distance is a symmetric function
• How to express the basic cognitive operations
  \(\text{asymmetric} \) similarity, \(\text{asymmetric} \) conjunction – in the geometric framework?
In the vector model, pairwise disjoint *possibilities* are represented by pairwise orthogonal subspaces.

In the simplest case 1-dimensional subspaces are represented by the axes of a Cartesian coordinate system.

A state is described by a vector $s$ of unit length.

The projections of $s$ onto the different axes are called *probability amplitudes*.

The square of the amplitudes are the relevant probabilities.

They sum up to 1 – the length of $s$:

$$ (P_a s)^2 + (P_b s)^2 = 1 $$
Projector to subspace $A$: $P_A$

Projected vector: $P_A \mathbf{s} = \mathbf{u}$

unique vector $\mathbf{u}$ such that $\mathbf{s} = \mathbf{u} + \mathbf{u}'$, $\mathbf{u} \in U$, $\mathbf{u}' \in U^\perp$.

The length of the projection is written $|P_A \mathbf{s}| = |\mathbf{u}|$.

The probability that $\mathbf{s}$ is about $A$ (that $\mathbf{s}$ collapses onto $A$) is the square of the length of the corresponding projection: $|P_A \mathbf{s}|^2$ (Born rule)
Order Dependence of Projections

\[ |P_a P_b s| \neq |P_a P_b P_a s| \]
Compatible and Incompatible Events

Compatible events are ones that may be assigned a simultaneous truth value. Thus, if event $X$ and event $Y$ are compatible, their conjunction $X \land Y$ is well defined. The probabilities for compatible events obey the Kolmogorov axioms. Two immediate consequences are that for compatible events $X$ and $Y$ we have,

$$p(X \land Y) = p(Y \land X),$$
$$p(X|Y) = p(Y|X) \frac{p(X)}{p(Y)}$$

In contrast with compatible events, incompatible ones are those for which $X \land Y$ is undefined. Thus although the probabilities $p^Q(X)$ and $p^Q(Y)$ exist, the joint $p^Q(X \land Y)$ may not. Typically one can define a modified version of conjunction with an explicit ordering, eg $X \land Y$ is taken to mean $X and then Y$ for incompatible variables. This implies that,

$$p^Q(X \land Y) \neq p^Q(Y \land X),$$
$$p^Q(X|Y) \neq p^Q(Y|X) \frac{p^Q(X)}{p^Q(Y)}.$$
Asymmetric Conjunction

- The sequence of projections \((P_a, P_b)\) corresponds to a Hermitian operator \(P_a P_b P_a\).

\[(P_a, P_b) = \text{def} \ P_a P_b P_a \quad (\text{Gerd Niestegge’s asymmetric conjunction})\]

- The expected probability for the sequence \((P_a, P_b)\) is

\[\mu(P_a, P_b) = |P_b P_a S|^2\]
Conditional Probabilities

- \( \text{Prob}(A|C) = \text{Prob}(CA)/\text{Prob}(C) \) (Classical)
- \( \text{Prob}(A|C) = \text{Prob}(CAC)/\text{Prob}(C) \) (Quantum Case, cf. Gerd Niestegge, generalizing Lüders' rule)
- If the operators commute, Niestegge's definition reduces to classical probabilities: \( CAC = CCA = CA \)
- Interferences
  - \( A = CA + C\downarrow A \) (classical, no interference)
  - \( A = CAC + C\downarrow AC\downarrow + CAC\downarrow + C\downarrow AC \) (interference terms)
Interference Effects

- Classical:
  \[ \text{Prob}(A) = \text{Prob}(A|C) \text{ Prob}(C) + \text{Prob}(A|\neg C) \text{ Prob}(\neg C) \]

- Quantum:
  \[ \text{Prob}(A) = \text{Prob}(A|C) \text{ Prob}(C) + \text{Prob}(A|C\perp) \text{ Prob}(C\perp) + \partial(C, A), \]
  where \[ \partial(C, A) = \text{Prob}(CAC\perp + C\perp AC) \] [Interference Term]

**Proof**

Since \( C + C\perp = 1 \), \( C\perp C = CC\perp = 0 \), we get
\[ A = CAC + C\perp AC\perp + CAC\perp + C\perp AC \]
Computing the Interference Term

- In the simplest case (when the propositions \( C \) and \( A \) correspond to projections of pure states) the interference term is easy to calculate:
  \[
  \vartheta(C, A) = \text{Prob}(CAC\perp + C\perp AC) \\
  = 2 \text{Prob}^{\frac{1}{2}}(C; A) \text{Prob}^{\frac{1}{2}}(C\perp; A) \cos \Delta
  \]

- The interference term introduces one free parameter: The phase shift \( \Delta \).
Solving the Tversky/Shafir Puzzle

- Tversky and Shafir (1992) show that significantly more students report they would purchase a nonrefundable Hawaiian vacation if they were to know that they have passed or failed an important exam than report they would purchase if they were not to know the outcome of the exam.

- \[ \text{Prob}(A/C) = 0.54 \]
  \[ \text{Prob}(A/C\perp) = 0.57 \]
  \[ \text{Prob}(A) = 0.32 \]

- \( \varphi(C, A) = \left[ \text{Prob}(A|C) \cdot \text{Prob}(C) + \text{Prob}(A|C\perp) \cdot \text{Prob}(C\perp) \right] - \mu(A) = 0.23 \)

\[ \Rightarrow \cos \Delta = -0.43; \Delta = 2.01 \approx 231° \]
Orthomodular Lattices

- The union of the two Boolean perspectives $F$ and $S$ gives an orthomodular lattice.

- The resulting lattice is non-Boolean. It violates distributivity: $\pi\{a\} \cup (\pi\{a\} \cap \pi\{d\}) = \pi\{a\} \cup \pi\{n\} = \pi\{b\}$

However, distributivity would result in $1$. 
Explaining Order Effects

• Order effects occur when the order of information influences judgments
  \[ P(E|X;Y) \text{ is not judged equal to } P(E|Y;X) \]

• It has also been shown that people commit what is known as the “inverse fallacy” that is, judging
  \[ P(X|Y) = P(Y|X) \]

• Both of these empirical findings are difficult to reconcile with an approach to causal reasoning based on Bayesian probability theory
Experiments:

- Participants were asked to make judgments involving conditional probabilities about various novel categories.
- Constructed a classical model and a QP model and used Bayesian analysis to compare them.
- Three main findings emerge:
  - 1) Both the classical and the QP models provide good fits for some sets of participants,
  - 2) The fits of the classical/QP models get better/worse as participants gain experience in the task,
  - 3) Participants who scored highest/lowest on the CRT task tended to be better fit by the classical/QP model.
Experiments:

Each animal had three binary features (X, Y, and E) where two of the features (X and Y) causally influenced the third (E) to form a common effect network.

For example, in the Lake Victoria Shrimp category, X = ACh neurotransmitter (high or low amount), Y = sleep cycle (accelerated or normal), and E = body weight (high or low).

Participants were given information about the typicality of feature values.

Cognitive Reflection Test (CRT) (Frederick, 2005), or may be influenced by experimental instructions, such as speed vs accuracy prompts.
Explaining Order Effects: Resource and Ability Constraints

- high = CRT score of 3, medium = CRT score of 1,2, and low = CRT score of 0.
- Order effects were assessed by comparing the judgments \( E|X1;Y2 \) and \( E|Y2;X1 \) and the judgments \( E|X2;Y1 \) and \( E|Y1;X2 \).
- For each individual, calculated an “order effect score” defined as \(|(E|X1;Y2)|||(E|Y2;X1)|+|(E|X2;Y1)|||(E|Y1;X2)|\)
  Higher scores indicate larger order effects and a more “quantum” representation of information.
- Participants were given 10 multiple-choice questions to make sure they understood the features and their relationships.
- After this test, participants completed six blocks of trials where they were asked to make judgments about the value of different features. There were two block types (BX and BY) that were repeated three times in an alternating fashion (e.g., BX, BY, BX, BY, BX, BY). Participants were randomly assigned to start with either the BX or BY block.
Results

Figure 1: Top panel: Order effect scores for three CRT groups across block pairs. Bottom panel: Inverse fallacy scores for three CRT groups across block pairs. Error bars show the standard error.
Results

Figure 2: Top panel: DIC values for the QP model as a function of block number for each of the CRT groups. Bottom panel: DIC values for the CP model as a function of block number for each of the CRT groups.
Results

• For the QP model the DIC values show the following patterns
  • First the model performs better in participant groups where the CRT score is lower
  • Second, while the model performance for the medium and high CRT groups does not vary much across blocks, performance clearly decreases with block number for the low CRT group

• For the classical model the opposite behavior is observed
  • First the model performs better in participant groups with a higher CRT score
  • Second, while performance for the high and medium CRT groups does not vary much across blocks, performance clearly increases with block number for the low CRT group

Jennifer Trueblood
Results

• Results show a clear separation between the order effect scores for the high CRT group versus the other two, suggesting that the high CRT group is more likely to be using a classical representation.
• Figure 1 also suggests that as participants gain experience with the scenarios the degree to which they display the (non-classical) order effect decreases.
• This suggests that participants’ representation of the events changes with experience to become gradually more classical and less quantum.
• The data also show an interaction between inverse fallacy score and CRT group.
• This provides evidence that the low CRT group is more likely using a QP representation.
Classical Information Processing

Cognitive system is in a **definite** state with respect to each possible measure

**Take a measure**
e.g. Similarity, Preference, Emotion, Memory

**Simply record**
what existed immediately before our measurement was taken

Jerome Busemeyer
Quantum Information Processing

Cognitive system is in an **indefinite** state with respect to each possible measure.

**Impose** a measure e.g. Similarity, Preference, Emotion, Memory

Create a definite state, bringing into existence a reality which was not there before.
Order Effects: Incompatible Representation

• Attitudes are not simply recorded and retrieved from memory and instead they are constructed online (Schwarz, 2007)

• The thoughts constructed from the first question change the context used for evaluating the second question, causing order effects

• Quantum theory provides an elegant way to formalize this intuition (Wang & Busemeyer, 2013; Wang, Solloway, Shiffrin, & Busemeyer, 2014).

70 national surveys from past 10 years

• 1) Are there robust order effects?
• 2) Is the QQ Equality satisfied?

• To examine this, obtained from the Pew and Gallup all its survey experiments in past 10 years, like the country satisfaction-presidential approval pair that used two question orders (not cherry-picking)

• 70 surveys; nationally representative samples of 651 -3,006 adults (M = 1,644, SD = 422).

• 2 lab exp.

Interference Effects and Paradoxes of Bounded Rationality

- **Conjunction and disjunction fallacy:** Aerts et al. (2005), Khrennikov (2006), Franco (2007), Conte et al. (2008), Blutner (2008), Busemeyer et al. (2011).

- **Prisoner’s dilemma:** Pothos and Busemeyer (2009).

- **Order effects:** Trueblood and Busemeyer (2012).

- **Categorization:** Aerts and Gabora (2005), Aerts (2009), Busemeyer, Wang, and Lambert-Mogiliansky (2009).
Recent Geometric (QM) Interest

- **Quantum Interaction** workshop at Stanford in 2007 organized by Bruza, Lawless, van Rijsbergen, and Sofge (as part of the AAAI Spring Symposium)
- **Workshops at Oxford** (England) in 2008,
- **Workshop at Vaxjo** (Sweden) in 2008,
- **Workshop at Saarbrücken** (Germany) in 2009,
- **AAAI meeting** Stanford in 2010.
- **Busemeyer** (Indiana University) et al. organized a special issue on Quantum Cognition, which was published in the October (2009), issue of Journal of Mathematical Psychology
- **Special Issues, Topics in Cognitive Science** (2013; 2014)
Summary of Simple QM

**Probability**

**Classical**
- Each unique outcome is a member of a set of points called the Sample space.
- Each event is a subset of the sample space.
- State is a probability function, $p$, defined on subsets of the sample space.

**Quantum**
- Each unique outcome is an orthonormal vector from a set that spans a Vector space.
- Each event is a subspace of the vector space.
- State is a unit length vector, $S$,

$$p(A) = \|P_A S\|^2$$
Summary of Simple QM Probability

Classical

• Suppose event A is observed (state reduction):
  \[ p(B \mid A) = \frac{p(B \cap A)}{p(A)} \]

• Commutative Property
  \[ p(B \cap A) = p(A \cap B) \]

Quantum

• Suppose event A is observed (state reduction):
  \[ p(B \mid A) = \frac{\|P_B P_A S\|^2}{\|P_A S\|^2} \]

• Non-Commutative
  \[ \|P_B P_A S\|^2 \neq \|P_A P_B S\|^2 \]
Outline

• New Probability/Logic Models in Information Retrieval

• New Probability Models in Human Inference and Cognition

• New Probability/Information Models in Distributed Inference and Control
Fundamental Problems

• *Interaction between information and control*
  – Controllers communication via “signaling strategies”
  – “information neighborhoods” for controllers
  – cost of information versus cost of control

• Despite pioneering work by Witsenhausen and others (formulations and results on the separation of the use of information (i.e., estimation) and control),
  – there does not exist today a satisfactory formulation of the joint “optimization” problem in information flow and control
  – Important to develop theories that treat control strategies and information patterns in a balanced manner
Fundamental Problems cont.

• **Concept of “state” for such a system not well understood** (Witsenhausen – some results towards resolving this problem)
  – In multi-agent stochastic problems there need not be a preassigned total time order of actions.
  – In fact, action times may depend on observations and controls by the same or other agents.

• Problem also related to the availability of a dynamic programming algorithm for solution
  – Important to develop local state descriptions.
  – Global state descriptions ~ centralized control
  – Local state models (local times), permit consideration of asynchronous actions by various controllers
Fundamental Problems cont.

- Interactions between measurements by different agents and between system dynamics and measurements
  - Akin to very strong interaction between information and control
  - Often the case where one cannot prove existence of an optimal control law (or design)
- Allow some flexibility over the information pattern
  - What can be said abstractly about the joint selection of information and control patterns?
- No strict preassigned order of action times for the various agents
Measurements Supporting Simple Propositions

- Everything is build on local measurements and local observables
- Simple events or propositions $E$ supported by databases in multi-sensor, multi-agent distributed systems
- Implication ($\leq$), orthocomplementation ($'$)
- Theorem 1: The set of simple propositions, in a distributed, asynchronous stochastic system is an orthomodular $\sigma$- orthoposet
- Structure of “logic” not sufficient – need models for statistics of “state” transitions
- States, local and global – just relevant variables – no memory interpretation
- Event-state structure ($E, S, P$)
Event-State Structures – Local and Global

- Consider probability function $P : E \times S \rightarrow [0, 1]$
- Event-state structure $(E, S, P)$
- Theorem 2: Given an event-state structure $(E, S, P)$ satisfying certain axioms compatible with general principles of information flow and databases in multi-agent systems, one can construct $(E, \leq, \cdot)$, and $\hat{S}$, such that
  - (a) $(E, \leq, \cdot)$ is an orthomodular $\sigma$-orthoposet.
  - (b) $\hat{S}$ is a strongly order-determining, $\sigma$-convex set of probability measures on $(E, \leq, \cdot)$.
  - (c) $\alpha \rightarrow \mu_{\alpha}$ is a bijection of $S$ onto $\hat{S}$
- The set $\hat{S}$ is dual to the set $S$
- Extend to local states for agents– communication effects
• **Theorem 3**: The databases in a multi-agent stochastic control system can be used to construct an event-state structure \((E, S, P)\) or \((E, \hat{S})\) as prescribed by Theorem 2. This representation is faithful in the sense that it can generate the statistics on which it was based.

• \(H\) a Hilbert space (complex); \(Q(H)\) the set of orthogonal projections with the usual order \(\leq\), and \(\perp\) being orthogonal complement

• \(S\) positive, trace class, self-adjoint operators on \(H\) with trace 1

• Let \(\hat{S} = \{\mu(\rho), \rho \in S; \mu(\rho) = Tr[\rho Q]\}\)

• \(\hat{S}, Q(H)\), event-state structure with \(P(Q, \rho) = Tr[\rho Q]\)

• \(\sigma\)-algebra \(E\) of subsets of \(X\) and \(\hat{S}\) a set of probability measures on \(X\)
Event-State-Operation Structures

- Multi-agent system – noncompatible events – occurrence cannot be verified by two or more agents
- Manifestation of communication constraints – or interactions
  - Sensor networks – domain of observation or sensor range
  - Multi-agent control – domain of influence or control range
- Incompatible measurements – new essential concept in multi-agent systems
- Need to build probabilistic models that have this incompatibility built-in – Conditioning and its modeling is at the center of this
- Multi-agent setting: data/measurements lead to two incompatible events – how do we describe their conjunction?
- Introduce generalizations of conditional probability and conditional expectation
Event-State-Operation Structures

- Introduce operations \( T : E \to \Sigma = \{ \text{maps from } S \text{ to } S \} \)
- \( T \) satisfies certain reasonable axioms
- For \( p \in E \), \( T_p \) is the operation corresponding to event \( p \)
- For \( \alpha \in S \), \( T_p\alpha \) is the new state conditioned on the occurrence of \( p \) and prior state \( \alpha \)
- Domain \( D_p \) of \( T_p \) are those states that can induce (influence) the occurrence of \( p \).
- The set of operations:
  \[
  \Sigma_T = \{ T_{p_1} \circ T_{p_2} \circ \cdots \circ T_{p_n} ; p_1, p_2, \ldots, p_n \in E \}
  \]
- We assert the existence of an event \( q_x \), that can be used to design experiments to determine whether or not a state belongs to the domain \( D_x \) of an operation
Event-State-Operation Structures

- $(\Sigma, \circ)_{T}$ is a multiplicative subsemigroup of $(\Sigma, \circ)$
- Introduce the $*$ operation on $\Sigma_{T}$ – reversal of application
- Show it is an involution
- A *Baer* semigroup* (Foulis, 1960), is an involution semigroup where the annihilator of each element is a principal left (right) ideal generated by a self-adjoint idempotent
- Theorem 4: If $(E, S, P, T)$ is an event-state-operation structure supported by the database and conditioning of a multi-agent stochastic control system, then $(\Sigma_{T}, \circ, *, \sim)$ is a Baer* semigroup
- $\sim$ is the map $x \rightarrow T_{q_{x}}$
Incompatible Events – Measurements

- Event-state structure ~ a **passive** picture
- Operations provide an **active** picture
- For $p, q \in E$, $p \leq q$ iff $T_p \circ T_q = T_p$ – “smoothing property” of conditional expectations

- Does $p \land q$ exist for $p, q \in E$?

- **Theorem 5**: If $(E, S, P, T)$ is an event-state-operation structure, then $(E, \leq, \cdot)$ is an ortholattice. Moreover, if $p, q \in E$, then $T_{p \land q} = (T_{p} \circ \bar{T}_{q}) \circ T_{q}$
Models with Incompatibility Build-in

- **Theorem 6**: Assumptions as in Theorem 5. Then for \( p, q \in E \), the following are equivalent:
  (a) \( p \) and \( q \) compatible
  (b) \( T_p \circ T_q = T_q \circ T_p \)
  If \( p \) and \( q \) are compatible \( T_{p \land q} = T_p \circ T_q \)

- **Active interpretation of operations**: can be thought of as a model for the combined operation of taking a measurement and applying a control law by the agent
- **Passive interpretation of operations**: system’s interaction to measurements (used by recent results in information retrieval systems)
- We also get an interpretation of the **conjunction of incompatible events or measurements** as “data fusion” or “agreement” between agents
Final Results and Future

- Classical systems ~ commutative Baer* semigroup
- Non-classical information patterns ~ noncommutative Baer* semigroups
- Structural properties of the system → structural properties of the semigroup?
- Needed: representation of Baer* semigroups, classification theory, coarse and fine structure theory
  - Baer* semigroup can be embedded in a C* algebra
  - By Gelfand-Naimark – there is a Hilbert space H s.t. the C* algebra is * isomorphic to a closed * subalgebra of $\mathcal{B}(H) = \text{the space of bounded operators on } H$

- Theorem 7: Databases and conditioning in a multi-agent stochastic control system can be used to construct an event-state-operation structure. This is a faithful representation.
Utility – Advantages

• Need to identify assumptions on the information pattern and flow that allow explicit representation of the Baer* semigroup
• Can result in simple examples
• Theorem 8: If \((E, \leq)\) is atomic and the atoms \(\hat{E}\) are mapped to pure states under \(T\), then \((E, S, P, T)\) can be represented as the lattice of projections on a Hilbert space.
• The **most useful ‘practical’ model**
  Finite dimensional Hilbert space, measurements to self-adjoint operators, states to trace one positive operators
• The **biggest payoff**:  
  Our theory (extensions of above) allow the formulation of ‘design’ problems as **convex problems over a pair of Banach spaces** (one for measurements, one for controls)
• Results **automatically to introduction of ‘supervisors’**
Optimization – Convex Structures

• Take the last example and consider unnormalized version of operations

• **Operation**: a positive linear map $T$ s.t.
  \[ 0 < \text{Tr}[T(\rho)] \leq \text{Tr}[\rho] \]

  Probability of transmission of a state $\rho$ by an operation $S$ is $\text{Tr}[S(\rho)]$

  Output state $\rho_{\text{out}} = S(\rho) / \text{Tr}[S(\rho)]$

• The Effect of an operation $S$, is the operation $A$ s.t.
  \[ \text{Tr}[S(\rho)] = \text{Tr}[\rho A] \]

• **Set of Effects** ($\text{EF}$) = Bounded operators $0 \leq A \leq 1$

• Projections are the extreme points of $\text{EF}$
Convex Structures

- Embed S into a real vector space V of functions on S
- \(\alpha \rightarrow P(., \alpha)\)
- Define a norm, consider the dual V* -- E is identified as a subset of V*
- Introduce a partial order in V by a cone V^+
- Can extend norm to a positive linear functional \(\tau : V \rightarrow \mathbb{R}, \quad |\tau(x)| \leq \|x\| \quad \text{for all } x \in V\)
- States S = \(\{x \in V^+ \mid \tau(x) = 1\}\) -- a convex set
- Effects \(\mathcal{EF} = \{\phi \in V^* \mid 0 \leq \phi \leq \tau\}\)

Convex, weak* compact, partially ordered, orthocomplementation \(\phi^\perp = \tau - \phi\)

Events E are the extreme points of \(\mathcal{EF}\)
Theorem 9: To any event-state-operation structure \((E, S, P, T)\), there corresponds a pair of Banach spaces \(V, V^*\), with positive cones and a trace functional \(\tau\) on \(V\). \(S\) is identified with a convex subset of \(V\). \(E\) as the extreme points of a convex subset of \(V^*\). Operations in \(\Sigma_T\) correspond to linear positive maps \(T : V \rightarrow V\), s.t. \(0 \leq \tau(T_x) \leq \tau(x)\) for all \(x \in V^+\). To every operation \(T\) we can associate its effect \(\phi_T\) via \(\tau(Tx) = \phi_T(x)\).
Complex Observation Processes

- Operation valued measures (OVM)
  
  A generalized sensor (or observation) on a measurable space \((U, B)\) is a map \(M : B \rightarrow \ell^+(V)\) such that
  
  (i) \(M(B) \geq M(\emptyset), \forall B \in \mathcal{B}\)

  (ii) for \(B_i \cap B_j = \emptyset, i \neq j\), \(M(\bigcup B_i) = \sum_{i=1}^{\infty} M(B_i)\)

  (iii) \(\tau(M(U)\rho) = \tau(\rho)\) for all \(\rho \in V\)

- For dynamic problems need families of generalized sensors – generalized stochastic processes

- Measurement \(K_M\) associated with \(M\) \((V^*\text{ valued measure})\)

- \(K_M(B)(\rho) = \tau[M(B)\rho]\)

- Composition \(M_{12}(B_1 \times B_2)(\rho) = M_2(B_2)M_1(B_1)\rho\)
Naimark’s Theorem

Measurement $K$ on product space $(U, \mathcal{B}) = (U_1, \mathcal{B}_1) \times (U_2, \mathcal{B}_2)$
Measure together two quantities in $U_1$ and $U_2$
Marginal measurements $M_1$ on $U_1$ and $M_2$ on $U_2$
If both $M_1$ and $M_2$ are projections and commute then $M_{12}$ is projection valued

Naimark's Theorem
Given a POM $M$ on $(U, \mathcal{B})$ with values in $\mathcal{L}(H)$
There exists a pure state $\rho_e$ on a Hilbert space $H_e$ and a PVM $E_M$ on $H \times H_e$
s.t. $\text{Tr}[\rho M(A)] = \text{Tr}[(\rho \times \rho_e) E(A)]$, for any $\rho$ and any $A \in \mathcal{B}$
Distributed M-ary Detection

• $N$ agents, with local observation times

$$(y^i(t_k^i), z^j(t_k^i)), \quad i = 1, \ldots, N, \quad j \neq i, \quad j = 1, \ldots, N, \quad k = 1, \ldots, L(i)$$

**Theorem 10:** In the distributed, $M$-ary detection problem, any sequence of observations and communications between the agents can be represented by an appropriate measurement $K_M(.)$ on some measurable space $(\mathcal{U}, \mathcal{B})$

• Can be used to obtain performance
• Does not produce the actual observation process and communication strategy
Distributed M-ary Detection

- $M$ hypotheses $H_1, \ldots, H_M$. Risk operators $W_i \in T_S(H)$
- Search for optimal measurement $K_M(.)$ subject to some information pattern constraint
- Let $\Pi_i = \int \Pi_i(u)K_M(du), \quad i = 1, \ldots, M$
- Problem: $\min \quad Tr \sum_{i=1}^{M} W_i \Pi_i$
- Over all positive operator valued measures (POM) $\Pi_i, 1, \ldots, M$ such that $\{\Pi_i\}_{i=1}^{M} \in A$
- Here $A$ is a convex set of POM’s corresponding to some information theoretic constraint on information (communication) patterns (e.g. capacity constraints)
• A convex linear programming problem! Leads to duality between decisions and information patterns.

• For the unconstrained problem we have

**Theorem 11**: Suppose the constraint set is the set of all POMs. Then a necessary and sufficient condition for the POM $\Pi^*_i$, $i = 1, \ldots, M$, to be optimal is

$$(i) \quad \sum_{j=1}^{M} W_j \Pi^*_j \leq W_i, \quad i = 1, \ldots, M$$

$$(ii) \quad \sum_{j=1}^{M} \Pi^*_j W_j \leq W_i, \quad i = 1, \ldots, M$$

Moreover the operator $Y = \sum_{j=1}^{M} W_j \Pi^*_j = \sum_{j=1}^{M} \Pi^*_j W_j$ is self-adjoint and is the unique solution of the dual problem.
Distributed M-ary Detection

• Easy to show that these are equivalent to $Y$ being self-adjoint and
  \[ W_i \geq Y, \quad i = 1, \ldots, M \]
  or
  \[ (W_i - Y) \Pi_i^* = \Pi_i^* (W_i - Y), \quad i = 1, \ldots, M \]
  and minimum is $\text{Tr} \ Y$

• Extended to Estimation and Recursive Filtering problems

• Implementation? How to realize $\Pi_i^*$ by a measurement process and a communication pattern

• $Y$ is a Lagrange multiplier – sensitivity with respect to information pattern constraints?
Quantum Logic and IF Logic

FOL

• For every x, there exists a y depending on x such that $B(x; y)$ is true \( \forall x \exists y B(x, y) \)

• For every x, there exists a y independent of x such that $B(x; y)$ is true \( \exists y \forall x B(x, y) \)

• For every x, and y, there exists a z independent from x and y such that $C(x; y; z)$ is true \( \exists z \forall x \forall y C(x, y, z) \)

• Dependencies and independencies which are not definable in FOL

• Henkin prefixes: For every x and z, there exists y depending only on x and w depending only on z such that $D(x; z; y; w)$ is true
Quantum Logic and IF Logic

• Signaling sequences: For every x there is a y which depends only on x and there is a z which depends only on y such that C(x; y; z) is true

• Languages which express FOL undefinable independencies
  • Henkin quantifiers (Henkin, 1959/1961)
  • Dependence between arbitrary objects (Fine 1983, 1985)
  • Independence-Friendly Logic (Hintikka and Sandu, 1989)
  • Dependence Logic (Väänänen, 2007)
  • Independence Logic (Graedel and Väänänen, 2009, Galliani, 2014, Abramski and Väänänen, 2009, etc)
  • Inclusion Logic (Galliani)
Quantum Logic and IF Logic

- Independence-Friendly Logic
- FOL + quantifiers and connectives of the form $(\exists x/W), (\forall x/W), (\lor/W), (\land/W)$
  with the interpretation: “the choice of $x$ is independent of the values of the variables in $W$”
- When $W = \emptyset$, we recover the standard quantifiers.

- Interpretation of IF Logic
- Let $X$ be a set of assignments (team) in a model $M$:
  - For $\psi$ a literal: $M, X \models \psi$ iff every assignment in $X$ satisfies $\psi$
Quantum Logic and IF Logic

• $M, X \models \varphi \lor \psi$ iff $M, Y \models \varphi$ and $M, Z \models \psi$
  for some $Y, Z$ such that $Y \cup Z = X$

• $M, X \models \varphi \land \psi$ iff $M, X \models \varphi$ and $M, X \models \psi$

• $M, X \models (\exists x/W)\varphi$ iff there is a $F : X \to M$
  such that $F$ is $W$-uniform and the team
  $X[x; F]$ which is formed by extending each
  assignment $s$ in $X$ with $(x, F(s))$ satisfies $\varphi$
  in $M$

• $M, X \models (\forall x/W)\varphi$ iff the team $X[x; M]$ formed
  by extending each assignment $s$ in $X$ with
  $(x, a)$ satisfies $\varphi$ in $M$.

• If $M, \{\varnothing\} \models \varphi$ we say that $\varphi$ is true in $M$. 
Quantum Logic and IF Logic

• Game-theoretical semantics for SENTENCES

Order of moves is determined by (a path in) the syntactical tree of \( \varphi \)

- Quantifier \( Qx \): choose an element \( a \) in \( M \)
  \[
  \begin{align*}
  Q &= \exists : \text{Verifier's move} \\
  Q &= \forall : \text{Falsifier's move}
  \end{align*}
  \]

- Binary connective \( o \): choose between immediate subformulas
  \[
  \begin{align*}
  o &= \lor : \text{Verifier's move} \\
  o &= \land : \text{Falsifier's move}
  \end{align*}
  \]

- Literal \( L(\bar{x}) \):
  \[
  \begin{cases}
  L(\bar{a}) \text{ true : Verifier wins} \\
  L(\bar{a}) \text{ false : Falsifier wins}
  \end{cases}
  \]

\[ M \models \varphi \text{ iff Verifier has a UNIFORM winning strategy} \]

• Signaling sequences (Hodges)
• Henkin sequences
• Coordination
• Quantum Logic is a fragment of IF Logic
• Games with partial information
Thank you!

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Questions?