The prices of packets and watts:
Optimal control of decentralized systems

P. R. Kumar
Dept. of Electrical and Computer Engineering
Texas A&M University

With Rahul Singh and Le Xie

Rahul Singh

Email: prk.tamu@gmail.com
Web: http://cesg.tamu.edu/faculty/p-r-kumar/
Happy 65\textsuperscript{th} Birthday, Demos!
Many fundamental contributions

- A Market-Based Approach to Optimal Resource Allocation in Integrated-Services Connection-Oriented Networks
- Distributed estimation algorithms for nonlinear systems
- Multi-armed bandit Problems
- Low-energy wireless communication network design
- Asymptotically efficient adaptive allocation rules for the multiarmed bandit problem with switching cost
- The decentralized Wald problem
- Stochastic routing in ad-hoc networks
- A market-based approach to optimal resource allocation in integrated-services connection-oriented networks
- Optimal control strategies in delayed sharing information structures
- Optimal design of sequential real-time communication systems
- …
Two problems

◆ Optimal scheduling of unreliable networks with hard end-to-end deadlines

◆ Optimal pricing and scheduling of generators and loads

◆ Both are stochastic distributed control problems

◆ Both have a solution based on “price”

Unreliable multi-hop networks with end-to-end delay constraints
Multi-hop network

- $F$ flows
- Flow $f$ has an end-to-end deadline $\tau_f$
- Let $r_f =$ packet delivery rate of flow $f$ (Timely throughput)
Nodal constraints

- Node $i$ has an average power constraint
  \[
  \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \left( \text{# of packets transmitted by node } i \text{ at time } t \right) \leq c_i
  \]

- Packet transmission succeeds with probability $p_{ij}$

- No interference: Directional antennas
Challenge of scheduling a distributed system

- Does optimal scheduling require knowledge of the complete network state?
- Obtaining network state instantaneously itself requires solving end-to-end delay problem
- Even if each node could obtain complete state, DP is intractable
  - Huge state space: \((V\Delta)^{F\Delta}\)
- Is optimal scheduling this distributed system difficult?

Transmit packet from Flow 2 since Flow 1 has downstream congestion

Congestion

No Congestion
Objective

\[ \text{Max } \sum_{f} \alpha_f r_f \]

Where: \( r_f = \text{Throughput of packets of flow } f \)

that have an end-to-end delay ≤ \( \tau_f \)

= Timely throughput of flow \( f \)

- The timely throughput of flow \( f \) is weighted by \( \alpha_f \)

- How to schedule the network?
Solution

- Constrained optimization problem over stationary randomized policies \( \pi \)

\[
\max \lim_{T \to \infty} \sup \pi \frac{1}{T} \sum_{t=1}^{T} \sum_{f} \alpha_{f} (\text{# of packets of flow } f \text{ delivered in time at time } t)
\]

Subject to

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} (\text{# of packets transmitted by node } i \text{ at time } t) \leq c_{i}
\]

- Lagrangian \( \mathcal{L}(\pi, \lambda) \)

\[
\max \lim_{T \to \infty} \sup \pi \frac{1}{T} \sum_{t=1}^{T} \left\{ \sum_{f} \alpha_{f} (\text{# of packets of flow } f \text{ delivered in time at time } t) \right\} + \sum_{i} \lambda_{i} c_{i}
\]

- Packet-by-Packet Decoupling

\[
\max \lim_{T \to \infty} \sup \pi \frac{1}{T} \sum_{f} \sum_{\text{Packets of flow } f \text{ released before time } T} \left\{ \alpha_{f} 1(\text{Packet is delivered on time}) - \sum_{i} \lambda_{i} 1(\text{Packet is transmitted by Node } i) \right\}
\]
Packet level decision making

\[ \alpha_i 1(\text{Packet is delivered on time}) - \sum_i \lambda_i 1(\text{Packet is transmitted by Node } i) \]

\[ V(d, t) = \alpha_f \text{ for all } t \geq 0 \]

\[ V(i, \tau) = \min \left\{ \lambda_i + p_{i,j} V(j, \tau - 1) + (1 - p_{i,j}) V(i, \tau - 1), V(i, \tau - 1) \right\} \]

- Packet solves Dynamic Program offline
- Easy to solve: Packet state space size is \( V\Delta \)
Optimal distributed solution

◆ The optimal solution completely decouples!

◆ Each packet makes decision to be transmitted or not, depending only on its own state (location, time to deadline)

◆ Optimal scheduling of a packet does not depend on
  – State of other nodes
  – State of other flows
  – Even other packets within its own flow!
How to obtain prices?

◆ If price $\lambda_i$ is too low
  – Too many packets ask to be transmitted
  – Average power $\lambda_i$ constraint is exceeded

◆ If price $\lambda_i$ is too high
  – Too few packets ask to be transmitted
  – Average power available $\lambda_i$ is not used

◆ Suggests tatonnement

$$\lambda_{i}^{n+1} = \lambda_{i}^{n} + \epsilon \ [\text{Power consumed by node } i - c_i].$$

◆ But even if price is exactly right, we will need to randomize some flow’s decisions to get the power to be exactly used up.
Dual Problem

- The Dual function is $D(\lambda) = \max_{\pi} \mathcal{L}(\pi, \lambda)$

- “Max” is attained by Single Packet Transportation Problem

- Dual Problem is $\max_{\lambda \geq 0} D(\lambda)$

- No Duality Gap, since can be reduced to LP
Optimality condition

- Suppose $\lambda^*$ is price vector
- $\pi(\lambda^*)$ optimal randomized policy for single-packet transportation problem for each flow $f$
- Suppose at every node $i$,
  - Either power constraint is satisfied with equality by $\pi(\lambda^*)$
    \[
    \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} (\text{# of packets transmitted by node } i \text{ at time } t) = c_i
    \]
  - Or $\lambda_i^* = 0$
- Then $\pi(\lambda^*)$ and $\lambda^*$ are optimal by Complementary Slackness
Combine Single-Packet Transportation Problems of all Flows

Use state-action probabilities

\[
\max \sum_{f \in F} \sum_{s=0}^{\tau_f} \sum_{i \in V} A_f \xi_f(i, d_f, s) p_{i, d_f}
\]

Reward

\[
\sum_{f \in F} \sum_{s=0}^{\tau_f} \sum_{i \neq j} A_f \xi_f(i, j, s) E \leq P_i \quad \forall i \in V,
\]

Power Constraint

\[
\sum_{j \in V, j \neq d_f} \xi_f(j, i, s) p_{j, i} + \sum_{m \in V} \xi_f(i, m, s)(1 - p_{i, m}) = \sum_{k \in V} \xi_f(i, k, s - 1) \quad \forall i \neq d_f, 1 \leq s \leq \tau_f
\]

Balance equations

\[
\sum_{j \in V} \xi_f(s_f, j, 0) = 1 \quad \forall f, \quad \xi_f(i, j, s) \geq 0
\]

Probabilities

- Very tractable LP solution: Low complexity
- Just $|V|^2 F \Delta$ variables, $|V| + |V| F \Delta + F + |V|^2 F \Delta$ constraints
- Reduction from exponential $(V \Delta)^F \Delta$ complexity
Near Optimality for Peak Power Constraint

- Suppose Node $i$ can only transmit $c_i$ packets concurrently

- Simply truncate at $c_i$
  - Similar to Whittle’s relaxation for restless bandits

- Theorem
  - Policy is asymptotically $O\left(\frac{1}{\sqrt{N}}\right)$ optimal as the total network capacity is scaled by $N$
Example: Explicit solution

- Optimal solution
- Optimal prices
  \[ \lambda = (0.068, 1.4, 0) \]
- Optimal solution
  \[
  \begin{align*}
  \pi^1(1, 3) &= 0.5, \pi^1(1, 2) = 0, \\
  \pi^1(2, 2) &= 1, \pi^1(2, 1) = 1, \\
  \pi^2(3, 3) &= 1/13, \pi^2(3, 2) = 0, \\
  \pi^2(2, 2) &= 1, \pi^2(2, 1) = 1.
  \end{align*}
  \]

Flow 1:
Node 1 transmit with probability 0.5 if time-till-deadline is 3, else drop

Flow 2:
Node 3 transmit with probability 1/13 if time-till-deadline is 3, else drop

Node 2: Both Flows Transmit
Example: Numerical Computation and Simulation Comparison

- Comparison with EDF-BP and EDF-SP
  - $A_1 = A_2 = 1$
  - $C_{(i,j)} = 1$
  - $\Delta_1 = \Delta_2 + 1$

Diagram:

- Flow 1
  - $p_{(1,2)} = 0.5$
  - $p_{(2,3)} = 0.5$
  - $p_{(2,5)} = 0.5$
  - $p_{(5,6)} = 0.5$
  - $\beta_1 = 1$
- Flow 2
  - $p_{(2,3)} = 0.5$
  - $p_{(3,4)} = 0.5$
  - $p_{(6,4)} = 0.5$
  - $\beta_2 = 1$

Graph:

- Timely Throughput
- Relative Deadline of Flow 2
- Optimal Policy
- EDF-BP
- EDF-SP
Remark

- Issues not considered here:
  - Contention, interference, coding
- Interesting development:
- 10.85 GHz spectrum in millimeter band released by FCC
  - 3.85 GHz licensed
  - 7 GHz unlicensed
  - Another 18GHz proposed
The Independent System Operator Problem in Power Systems
The ISO problem in a simple static context

- ISO has to balance supply and demand
- Generator and load bid their supply and demand curves
- ISO intersects to find right price
Uncertainties and dynamics in the era of renewables and demand response

- Dynamic constraints: ramping, thermal inertia
- Uncertainty: Wind, temperature, water flow
- All choices have costs/benefits
- How can ISO ensure maximum social welfare?
- How much should be generated, and balance
- How should generators and loads bid?
The ISO problem

ISO

\[
\text{Max } E \sum_{i=1}^{N} \sum_{t=0}^{T} U_i(x_i(t), u_i(t))
\]

Balance: \[\sum_{i=1}^{N} u_i(t) = 0 \text{ for all } t\]

How to assign \(u_i(t)\)'s?

Without knowledge of:
- States \(x_i(t)\)
- Models \(f_i(x_i,u_i,w_i)\)
- Utilities \(U_i(x_i,u_i)\)
Coordinating multiple deterministic dynamical systems

- All generators, loads, storage and prosumers are deterministic dynamical systems

\[
x_i(t + 1) = f_i(x_i(t), u_i(t))
\]

\[
(x_i(t), u_i(t)) \in G_i(t)
\]

- Goal

\[
\text{Max} \sum_{i=1}^{N} \sum_{t=0}^{T} U_i(x_i(t), u_i(t))
\]

s.t. \( \sum_{i=1}^{N} u_i(t) = 0 \) for all \( t \)
The solution

◆ Lagrange Dual

\[
D(p) \triangleq \max_{u} \left( \sum_{t=0}^{T} \sum_{i=1}^{N} U_i(x_i(t), u_i(t)) - \sum_{t=0}^{T} p(t) \sum_{i=1}^{N} u_i(t) \right)
\]

\[
= \sum_{i=1}^{N} \max_{u_i} \left( \sum_{t=0}^{T} U_i(x_i(t), u_i(t)) - p(t)u_i(t) \right)
\]

Decomposes

◆ ISO needs to announce \( p = (p(0), p(1), \ldots, p(T)) \)
  – The entire sequence of prices for all future times

◆ Agent \( i \) distributedly chooses \( u_i = (u_i(0), u_i(1), \ldots, u_i(T)) \) to maximize its own utility

\[
\sum_{t=0}^{T} \left( U_i(x_i(t), u_i(t)) - p(t)u_i(t) \right)
\]
How does ISO choose price sequence $p$?

- The ISO needs to solve dual problem: $\min_{p \geq 0} D(p)$

- Subgradient iteration $p^{k+1} = p^k - \varepsilon^k \frac{\partial D(p^k)}{\partial p^k}$

- Now $\frac{\partial D}{\partial p} = -\left( \sum_{i=1}^N u_i(0), \sum_{i=1}^N u_i(1), ..., \sum_{i=1}^N u_i(T) \right)$

- Price iteration $p^{k+1} = p^k + \varepsilon^k \left( \sum_{i=1}^N u_i^k(0), \sum_{i=1}^N u_i^k(1), ..., \sum_{i=1}^N u_i^k(T) \right)$

- Converges after weighted averaging under convexity and compactness assumptions ("ergodic" method)
Convergence?

- Simplified version of Pravin’s Example
- Social welfare problem is
  \[
  \text{Min} \left[ -\frac{2}{5} u_1 - \frac{1}{10} u_1 - \log(1+u_2) \right]
  \]
  Subject to: 
  \[-1 \leq u_1 \leq 0 \]
  \[0 \leq u_2 \leq 2\]
  \[u_1 + u_2 = 0\]

- Optimal solution is \((u_1^*, u_2^*) = (-1, 1)\) and \(\lambda^* = \frac{1}{2}\)

- But even if ISO declares the optimal price \(\lambda^* = \frac{1}{2}\), generator is indifferent to -1 or 0, and could declare 0

- So how can the ISO determine the correct price?
The “averaging” solution

◆ Gustavsson et al 2015: Take $\theta \geq 0$

$$\bar{u}_i^k = \frac{1}{k} \sum_{s=1}^{k-1} s^\theta \bar{u}_i^{k-1} + \frac{k^\theta}{\sum_{s=1}^k s^\theta} u_i^k; \quad \bar{u}_i^0 = u_i^0$$

◆ Then $\bar{u}_i^k \rightarrow u_i^*$ and $\lambda_k \rightarrow \lambda^*$

◆ In this example:

$$\{\lambda_k\} : 0, 0.1, 0.6667, 0.5416, \ldots \rightarrow \frac{1}{2}$$

$$\{u_k\} : \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -0.94 \\ 0.12 \end{bmatrix}, \begin{bmatrix} -0.99 \\ 0.41 \end{bmatrix}, \begin{bmatrix} -0.997 \\ 0.73 \end{bmatrix}, \ldots \rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
Convergence

◆ General result

\[
\min \sum_{i=1}^{M} [c_i(u_i) + e_i(u_i)]
\]

\[
F_iu_i \leq g_i, h_i(u_i) \leq 0
\]

\[
\sum_{i=1}^{M} E_iu_i = d
\]

◆ \(c_i, e_i, h_i\) convex, compactness, optimal exists

◆ Slater’s condition

◆ Then convergence after averaging
In the deterministic case:
Price sequences and Bid sequences

- Nuclear power plant
  - \((p^2(0), \ldots, p^2(T))\)

- Coal power plant
  - \((p^2(0), \ldots, p^2(T))\)

- Wind farm
  - \((p^2(0), \ldots, p^2(T))\)

- Hydropower plant
  - \((p^2(0), \ldots, p^2(T))\)

- ISO

- Load serving entity
  - \((p^2(0), \ldots, p^2(T))\)

- Commercial load
  - \((p^2(0), \ldots, p^2(T))\)

- Industrial load
  - \((p^2(0), \ldots, p^2(T))\)

- Storage service
  - \((p^2(0), \ldots, p^2(T))\)
...... until convergence
Coordinating multiple stochastic dynamical systems with common uncertainty

- Generators, loads, storage, prosumers are stochastic – dependent on a common uncertainty
- Ex: All loads depend on common temperature of the city

\[ x_i(t + 1) = f_i(x_i(t), u_i(t), w(t)) \]
\[(x_i(t), u_i(t)) \in G_i(t) \]

- Common uncertainty \( w(\cdot) \) is observed by all loads/gens

- Goal \( \max E_{w(\cdot)} \left( \sum_{i=1}^{N} \sum_{t=0}^{T} U_i(x_i(t), u_i(t)) \right) \)
Uncertainty tree

- Uncertainty at each stage
- Tree policy: Choose action $u(v)$ for each vertex $v$
- Optimization problem

$$\min \sum_v p(v)c(x(v,u^v),u^v)$$
$$\text{s.t. } \sum_{i=1}^N u_i(v) = 0 \text{ for all } v$$

- At time 0, bid-price iteration for all nodes until convergence
  - Only needs to be done at time 0
- Very complex since number of loads is exponentially large
The stochastic case with private uncertainties

- Generators, loads, storage and prosumers are stochastic having private uncertainties

\[
\begin{align*}
x_i(t + 1) &= f_i(x_i(t), u_i(t), w_i(t)) \\
x_i(t) &\in X_i(t) \\
u_i(t) &\in U_i(t)
\end{align*}
\]

- Goal: \[\text{Max } E_{w_1(\cdot), w_2(\cdot), \ldots, w_N(\cdot)} \left( \sum_{i=1}^{N} \sum_{t=0}^{T} U_i(x_i(t), u_i(t)) \right)\]
Bid-Price iteration for private stochastic case

- At time 0: bid-price iteration for all nodes
- At time 1: bid-price iteration for subtree
- …
- ISO needs to know and announce
  - *Labels* of remaining subtree
- Agents need to know
  - *Laws* of all labels
  - Labels could be hashed for confidentiality
- Agent $i$ communicates *label* of $w_i(t)$ to ISO at time $t$
- Same complexity as before
- Also repeated at each stage
- Not entirely satisfactory
A tractable solution: Generators and loads modeled as LQG systems
Multiple LQG Systems with private uncertainty

- All generators and loads are LQG systems having private uncertainties

\[
x_i(t + 1) = A_i x_i(t) + b_i u_i(t) + C_i w_i(t)
\]

\[
y_i(t) = D_i x_i(t) + H_i v_i(t)
\]

\[
x_0, w_i(t), v_i(t) \sim N, \text{ mean 0, and independent}
\]

- Goal: \[
\text{Max } E \left( \sum_{i=1}^{N} \sum_{t=0}^{T} x_i(t)^T Q_i x_i(t) + u_i(t)^T R_i u_i(t) \right)
\]
Scheme for distributed LQG systems

◆ At each time \( t = 0, 1, 2, \ldots \)
  
  – Iterations \( k=1,2,3,\ldots: \)
    
    » ISO announces future price sequence \((p^k(t), \ldots, p^k(T))\)
    
    \[
    p^k = p^{k-1} + \varepsilon_{k-1} \left( \sum_{i=1}^{N} u_i^{k-1}(t), \sum_{i=1}^{N} u_i^{k-1}(t+1), \ldots, \sum_{i=1}^{N} u_i^{k-1}(T) \right)
    \]
    
    » Each Generator/Load \( i \) responds with optimal solution \((u_i^k(t), u_i^k(t+1), \ldots, u_i^k(T))\) of deterministic LQ problem
    
    \[
    x_i(t+1) = A_i x_i(t) + b_i u_i(t)
    \]
    
    \[
    \text{Min} \left( \sum_{i=0}^{T} x_i(t)^T Q_i x_i(t) + u_i(t)^T R_i u_i(t) + p^k(t)u_i(t) \right)
    \]

◆ Adequate number of iterations at each stage
Future price sequences and Future bid sequences at each time t

- Nuclear power plant
  - (p₁(t),...,p₁(T))

- Coal power plant
  - (p₁(t),...,p₁(T))

- Wind farm
  - (p₁(t),...,p₁(T))

- Hydropower plant
  - (p₁(t),...,p₁(T))

- ISO
  - (p₁(t),...,p₁(T))

- Load serving entity
  - (p₁(t),...,p₁(T))

- Commercial load
  - (p₁(t),...,p₁(T))

- Industrial load
  - (p₁(t),...,p₁(T))

- Storage service
  - (p₁(t),...,p₁(T))
Future price sequences and Future bid sequences at each time $t$

- Nuclear power plant
- Coal power plant
- Wind farm
- Hydropower plant
- ISO
- Load serving entity
- Commercial load
- Industrial load
- Storage service

$(u^1(t),\ldots,u^1(T))$
Other electrical network constraints: Power flow, etc

Can handle multiple linear equalities

\[ h_{11}(t)u_1(t) + \ldots + h_{1N}u_N(t) = \alpha_1(t) \]

\[ \ldots \]

\[ h_{m1}(t)u_1(t) + \ldots + h_{mN}u_N(t) = \alpha_m(t) \]
Remarks

- With 15 min intervals, is such an iterative bidding feasible?
- Even if correct prices are known, we still need “exploration” to determine allocations
- Is there a better alternative to get to optimal social welfare?
- Should we get to social welfare?
- Leakage of information
- Ongoing work
  - Strategic considerations in bidding
- Many other issues
  - Line limits
  - Line losses
  - AC power flow equations
  - Security constrained OPF
Thank you and wish you all the best, Demos and Barbara, in the coming years and decades.