Reaching Rational Expectations Equilibrium

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Speculative Market

- Two agents can **buy/sell** claim for an asset at price $p$
- Value of the asset, $X$, is random
- Ex-post profit of agent $i$:
  $$J^i = (X - p)y^i, \quad y^i = 1(\text{buy}) \text{ or } y^i = -1(\text{sell})$$
- Market clears when $\sum_i y^i = 0$
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- Market clears when $\sum_i y^i = 0$
- What is the equilibrium price?
Virtual Bidding: Traders place buy and sell bids in day ahead market (DAM), together with the reverse bid in the real time market (RTM).

Buy bids form a demand function $D(p)$, sell bids form a supply function $S(p)$
- $D(p)$: Total MW buy bids at price $\geq p$
- $S(p)$: Total MW sell bids at price $\leq p$

VB market price, $p^*$, is determined by $D(p^*) = S(p^*)$

RTM random price is $X$

Trader who bought the VB at $p^*$ (buyer) must sell it at price $X$
- Profit of the buyer is $(X - p^*)$. Similarly, profit of the seller is $(p^* - X)$
Common Prior and Private Signals

- \((\Omega, \mathcal{F})\) is a measurable space, \(\nu\) is a measure on \(\Omega\)
- Each agent has the same prior \(\nu\)
- Agent \(i\) receives a private signal \(S^i\) before the trade
- \(X(\omega), S^1(\omega)\) and \(S^2(\omega)\) are random variables
- Each agent wants to maximize her *expected profit*
Common Prior and Private Signals

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- \(X(\omega), S^1(\omega)\) and \(S^2(\omega)\) are random variables
- Each agent wants to maximize her \textit{expected profit}
- What is the appropriate notion of equilibrium price?
Private Information Equilibrium

- Price taking agents

\[ y^i \in \arg \max_{y^i \in \{1, -1\}} J^i(y^i) = \mathbb{E} [(X - p)y^i|S^i] \]

- Agent \( i \) will decide to:

\[
\begin{align*}
\text{Buy} & \quad \text{if} \quad \mathbb{E} [X|S^i] \geq p \\
\text{Sell} & \quad \text{if} \quad \mathbb{E} [X|S^i] < p
\end{align*}
\]

Private Information Equilibrium

Suppose \( p^{pie} = p^{pie}(\omega) \) is such that for every \( \omega \in \Omega \) the market clears when each agent's decision follows (1). Then \( p^{pie} \) is a private information equilibrium.
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  Buy if \( \mathbb{E} [X | S^i] \geq p \)

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(1)

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- Private information equilibrium always exists
Private Information Equilibrium: Example

- Signal $S^i \in \{1, 2, 3\}, \ i = 1, 2$
- Signals are i.i.d. : $\nu(S^1 = m, S^2 = n) = \frac{1}{9}, \ \forall m, n \in \{1, 2, 3\}$

![Diagram](image_url)

$X(\omega)$
Private Information Equilibrium: Example

- Signal $S^i \in \{1, 2, 3\}$, $i = 1, 2$
- Signals are i.i.d.: $\nu(S^1 = m, S^2 = n) = \frac{1}{9}$, $\forall m, n \in \{1, 2, 3\}$

\[
\begin{array}{c|c|c|c}
S^2 = 3 & \text{Agent 1 Sells} & \text{Agent 1 Buys} \\
S^2 = 2 & & \\
S^2 = 1 & & \\
\hline
S^1 = 1 & 0 & \frac{1}{2} & \frac{1}{2} \\
S^1 = 2 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\
S^1 = 3 & \frac{1}{2} & \frac{2}{3} & \frac{2}{3} \\
\end{array}
\]

- $p^{pie} = \frac{(E[X|S^1] + E[X|S^2])}{2}$
- There can be more than one equilibrium
Each agent sends her bids, $b^i = \mathbb{E}[X | S^i]$, to the auctioneer.
Private Information Equilibrium: Implementation

- Each agent sends her bids, $b^i = \mathbb{E}[X|S^i]$, to the auctioneer.
- Auctioneer announces the price, $p$. 

Diagram:
- Auctioneer
  - $p$ to Agent 1
  - $p$ to Agent 2
Private Information Equilibrium: Implementation

- Each agent sends her bids, $b^i = \mathbb{E}[X|S^i]$, to the auctioneer.
- Auctioneer announces the price, $p$.
- Transactions are made.

![Diagram showing the flow of bidding from agents to the auctioneer and the announcement of the price.](image-url)
Information from Prices

- Market clearing price \( p^* \) contains information about traders’ private signal
- Traders may wish to revise their trade after \( p^* \) is announced
  - Private information equilibrium is not ‘fully rational’
- More precisely, suppose
  - \( p_1^* \) is a PIE price. Trades are made, market clears
  - Market reopens. Each agents is asked to submit her bid again
  - Each agent has a new demand (supply) function \( D(p; p_1^*) \) \( (S(p; p_1^*)) \)
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  - Will there be a trade? Will there be a new equilibrium price $p_2^*$?
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  - Will there be a trade? Will there be a new equilibrium price $p_2^*$?
  - What does the price sequence $p_1^*, p_2^*, \cdots$ converge to?
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- We need a different equilibrium notion
  - Agents should be able to make rational expectations from the price $p^{**}$
  - There should be no need for re-trading even if the market reopens
Can agents infer about the state of the world ($\omega$) from the prices?
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In the private information equilibrium example:

- When $S^1 = 2$, $S^2 = 3$, $b^1 = 2/3$, $b^2 = 1/3$
- $p^{pie} = 1/2$ and agent 1 is the *buyer*
- But from $p^{pie} = 1/2$ agent 1 knows that $b^2 < 2/3$
- Then agent 1 can infer that $S^2 = 3$. In turn infers that $X = 0$
- Agent 1 will *sell* if the market reopens
Information from Prices: Example

- Can agents infer about the state of the world ($\omega$) from the prices?
- In the private information equilibrium example:
  - When $S^1 = 2$, $S^2 = 3$, $b^1 = 2/3$, $b^2 = 1/3$
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  - Then agent 1 can infer that $S^2 = 3$. In turn infers that $X = 0$
  - Agent 1 will sell if the market reopens
- Agents are more sophisticated: private information equilibrium is inadequate
- What is the appropriate equilibrium notion?
Rational Expectation Equilibrium (REE)

- Prices and information from prices:
  \[ p^{ree}(\cdot) : \Omega \rightarrow \mathbb{R}, \quad S(p) = \{ \omega \in \Omega : p^{ree}(\omega) = p \} \]

- Each agent will maximize the expected profit
  \[ J^i(y^i) = \mathbb{E} \left[ (X - p)y^i | S^i, S(p) \right] \]

- Agent \( i \) will decide to:
  \[ \begin{align*}
    \text{Buy} & \quad \text{if } \mathbb{E} \left[ X | S^i, S(p) \right] \geq p \\
    \text{Sell} & \quad \text{if } \mathbb{E} \left[ X | S^i, S(p) \right] \leq p 
  \end{align*} \]  
  \hspace{1cm} (2)
Rational Expectation Equilibrium (REE)

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- Each agent will maximize the expected profit
  \[ J^i(y^i) = \mathbb{E} [(X - p)y^i | S^i, S(p)] \]

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  \end{align*}

Rational Expectations Equilibrium

Suppose \( p^{\text{ree}} = p^{\text{ree}}(\omega) \) is such that for every \( \omega \in \Omega \) the market clears when each agent's decision follows (2). Then \( p^{\text{ree}} \) is a rational expectation equilibrium.
Suppose $S^1 = S^2 = 3$. Then agents know $p = 1/2$

Then, agent 1 can infer that $S^2 \in \{2, 3\}$. So, $E[X|S^1, S(p)] = 1/2$

Similarly, agent 2 can infer that $S^1 \in \{2, 3\}$. So, $E[X|S^2, S(p)] = 1/2$
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Expected profit of each agent is zero!
No Trade Theorems

Theorem (Tirole (1982), Milgrom and Stokey (1982))

In an REE of a purely speculative market with risk-averse or risk-neutral traders, risk-averse traders do not trade; risk-neutral traders may trade, but they do not expect any gain from their trade.

- Under standard assumptions in economics of a common prior and Bayesian rationality, there cannot be purely speculative transactions.
Convergence to REE

- Does there always exist an REE? Are there multiple REEs?
- How do the agents find an REE?
  - Agents can exchange bids/information until they agree on a price
  - Does such an iteration always converge?
Convergence to REE

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Asymptotic Agreement

- Two agents trying to estimate the value of a random variable
  - At step 1, agent 1 computes $A_1^1 = \mathbb{E}[X|S^1]$
    Sends $A_1^1$ to agent 2
  - Agent 2 computes $A_1^2 = \mathbb{E}[X|\mathcal{F}_1^2]$, $\mathcal{F}_1^2 = \sigma(S^2, A_1^1)$
    Sends $A_1^2$ to agent 1
  - At step $k$, agent 1 computes $A_k^1 = \mathbb{E}[X|\mathcal{F}_k^1]$, $\mathcal{F}_k^1 = \sigma(S^2, A_1^2, \ldots, A_{k-1}^2)$
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- Objective was distributed estimation and prediction. See Delphi method, predictive markets, wisdom of crowds.
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**Theorem (Borkar and Varaiya (1982))**

$A_k^i$ converges to a random variable $A_\infty^i$ a.s. Moreover, $A_\infty^1 = A_\infty^2$. 
Asymptotic Agreement

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    Sends $A^2_1$ to agent 1
  - At step $k$, agent 1 computes $A^1_k = \mathbb{E}[X|\mathcal{F}^1_k], \mathcal{F}^1_k = \sigma(S^2, A^2_1, \ldots, A^2_{k-1})$
    Sends $A^1_k$ to agent 2

- Objective was distributed estimation and prediction. See Delphi method, predictive markets, wisdom of crowds.

**Theorem (Borkar and Varaiya (1982))**

$A^i_k$ converges to a random variable $A^i_\infty$ a.s. Moreover, $A^1_\infty = A^2_\infty$.

- This iteration indeed converges to an REE!
  - In a pure speculative market with risk-neutral agents, REE always exists.
A Different Market Implementation

- Agents may be unwilling to fully reveal their assessments.
A Different Market Implementation

- Agents may be unwilling to fully reveal their assessments.
- A less revealing implementation of REE
  - At step 1, each agent $i$ computes $A^i_1 = \mathbb{E}[X|S^i]$  
    Sends this bid to the auctioneer
  - Auctioneer computes $M_1 = \mathbb{I}\{A^1_1 < A^2_1\} - \mathbb{I}\{A^1_1 > A^2_1\}$  
    Broadcasts this to both agents
  - At step $k$, each agent $i$ computes $A^i_k = \mathbb{E}[X|\mathcal{F}^i_k]$,  
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- Note that bids are NOT revealed to other agents
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**Theorem (Kalathil and Varaiya (2016))**

$A^i_k$ converges to $A^i_\infty$ a.s. Also, $A^1_\infty = A^2_\infty = P$. Moreover $P$ is an REE.
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**Theorem (Kalathil and Varaiya (2016))**

$A^i_k$ converges to $A^i_\infty$ a.s. Also, $A^1_\infty = A^2_\infty = P$. Moreover $P$ is an REE.

- Key idea is to show that $P$ is a common information
Prices, Price Expectations and Common Knowledge

- Suppose $p : \Omega \rightarrow \mathbb{R}$ is the true price function
- Agents believe that $q : \Omega \rightarrow \mathbb{R}$ is the price function (price expectations)
- What is the equilibrium notion?
  - Radner equilibrium
Suppose $p : \Omega \rightarrow \mathbb{R}$ is the true price function.

Agents believe that $q : \Omega \rightarrow \mathbb{R}$ is the price function (price expectations).

What is the equilibrium notion?

- Radner equilibrium

Rational expectations equilibrium is a Radner equilibrium if price expectations, $q(\cdot)$, is the same as the price function, $p(\cdot)$.

- Traders understand the true relationship between hidden information and prices.
Extension to $N$ agents

- At step $k$, each agent submits their bid $A_k^i = \mathbb{E}[X|\mathcal{F}_k^i]$ to the auctioneer.
- Auctioneer computes the median $M_k$ of $\{A_k^1, \ldots, A_k^N\}$.
- Auctioneer publishes the list of agents who are below and above the median.
- Note that the bids $A_k^i$'s are not published.
Extension to $N$ agents

- At step $k$, each agent submits their bid $A_{ik} = \mathbb{E}[X | \mathcal{F}_k^i]$ to the auctioneer
- Auctioneer computes the median $M_k$ of $\{A_{1k}, \ldots, A_{Nk}\}$
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**Theorem (Kalathil and Varaiya (2016))**

$A_{ik}$ converges to $A_{i\infty}$ a.s. Also, $A_{i\infty} = A_{j\infty} = P, \forall i, j$. Moreover $P$ is a REE.
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**Theorem (Kalathil and Varaiya (2016))**

$A^i_k$ converges to $A^i_\infty$ a.s. Also, $A^i_\infty = A^j_\infty = P, \forall i, j$. Moreover $P$ is a REE.

- Open question
  - Is it sufficient to send $M^i_k = \mathbb{I}\{A^i_k < M_k\} - \mathbb{I}\{A^i_1 > M_k\}$?
  - Each agent knows whether her bid is above or below the median
Convergence of REE: Example 1

- Agents exchanges the bids. Agent 1 starts first

- Assume $S^1 = S^2 = 3$
- Agent 1 computes $A^1_1 = \mathbb{E}[X|S^1] = 2/3$
  Sends this to agent 2
- Agent 2 infers that $S^1 \in \{2, 3\}$
- Agent 2 computes $A^2_1 = \mathbb{E}[X|\mathcal{F}^2_1] = 1/2$
  Sends this to agent 1
- Agent 1 infers that $S^2 \in \{2, 3\}$.
- Agent 1 computes $A^1_2 = \mathbb{E}[X|\mathcal{F}^1_2] = 1/2$
  Sends this to agent 2
- Agent 2 infers that $S^1 \in \{2, 3\}$
- Converges to $p^{ree}(S^1 = 3, S^2 = 3) = 1/2$
Convergence of REE: Example 2

- Agents exchanges the bids. Agent 2 starts first.

- Assume $S^1 = S^2 = 3$
- Agent 2 computes $A^2_1 = \mathbb{E}[X|S^2] = 1/3$. Sends this to agent 1
- Agent 1 infers that $S^2 = 3$
- Agent 1 computes $A^1_1 = \mathbb{E}[X|\mathcal{F}^1_1] = 1$ Sends this to agent 2
- Agent 2 infers that $S^1 = 3$
- Agent 3 computes $A^2_2 = \mathbb{E}[X|\mathcal{F}^2_2] = 1$
- Converges to $p^{ree}(S^1 = 3, S^2 = 3) = 1$
Convergence of REE: Example 2

- Agents exchanges the bids. Agent 2 starts first.

- Assume \( S^1 = S^2 = 3 \)
- Agent 2 computes \( A^2_1 = \mathbb{E}[X|S^2] = 1/3 \).
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- Agent 1 infers that \( S^2 = 3 \)
- Agent 1 computes \( A^1_1 = \mathbb{E}[X|\mathcal{F}_1^1] = 1 \)
  Sends this to agent 2
- Agent 2 infers that \( S^1 = 3 \)
- Agent 3 computes \( A^2_2 = \mathbb{E}[X|\mathcal{F}_2^2] = 1 \)
- Converges to \( p^{ree}(S^1 = 3, S^2 = 3) = 1 \)

- There exists more than one REE
Convergence of REE: Example 3

- Agents send their bids to the auctioneer

- Assume $S^1 = S^2 = 3$.
- $A^1_1 = \mathbb{E}[X|S^1] = 2/3$, $A^2_1 = \mathbb{E}[X|S^2] = 1/3$
- Auctioneer broadcasts
  $M_1 = \mathbb{I}\{A^1_1 < A^2_1\} - \mathbb{I}\{A^1_1 > A^2_1\} = -1$
- Agent 1 infers that $S^2 = 3$
- Agent 2 infers that $S^1 \in \{2, 3\}$
- $A^1_2 = \mathbb{E}[X|\mathcal{F}^1_2] = 1$, $A^2_2 = \mathbb{E}[X|\mathcal{F}^2_2] = 1/2$
- Auctioneer broadcasts
  $M_2 = \mathbb{I}\{A^1_2 < A^2_2\} - \mathbb{I}\{A^1_2 > A^2_2\} = -1$
- Agent 2 infers that $S^1 = 3$
- Converges to $p^{ree}(S^1 = 3, S^2 = 3) = 1$
Full Communication Equilibrium and REE

- Full communication equilibrium if $p^{fce} = \mathbb{E}[X|S^1, S^2]$
- Full communication equilibrium is an REE
- Generically, REEs are full communication equilibria
  - Suppose $X$ is given. $Q = \{\nu : \text{there exists a } p^{ree}, p^{ree} \neq p^{fce}\}$
  - Lebesgue measure of $Q$ is zero
- In the above example, $\nu$ is of dimension 8
- However, $\nu$ supports a $p^{ree}, p^{ree} \neq p^{fce}$, which satisfies
  $$\sum_{s^2 \in \{2, 3\}} \nu(s^2|s^1 = 3, s^2 \in \{2, 3\})X(3, s^2) = \sum_{s^1 \in \{2, 3\}} \nu(s^1|s^2 = 3, s^1 \in \{2, 3\})X(s^1, 3)$$
- So, $Q$ has a smaller dimension
Rational Expectations vs Consensus Algorithms

- **Consensus Algorithm**
  - Each agent has an initial estimate, \( b_0^i = \mathbb{E}[X|S^i] \)
  - Agents exchange the current estimate \( b_k^i \) with their neighbors
  - Agents update their estimates:
    \[
    b_{k+1}^i = \sum_{j \in N_i} w_{ij} b_k^j.
    \]
  - Convergence: \( b_k \to b^* \)
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  - Agents update their estimates:
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    b^i_{k+1} = \sum_{j \in \mathcal{N}_i} w_{ij} b^j_k. \quad b_{k+1} = Wb_k.
    \]
  - Convergence: \( b_k \to b^* \)

- Agents are *not rational*, not making any inference
  - Passive computation according to a predetermined function
  - \( b^* = f(b^1_0, b^2_0, \ldots, b^N_0) \), function of the initial estimates
  - Possible method to converge to a PIE in the absence of an auctioneer
Agents with Different Priors

- Agents have different priors
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  - If the (different) priors are common knowledge, agents can still interpret the messages consistently
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Agents with Inconsistent Priors

- $(\Omega, \mathcal{F})$ be a measurable space and $\nu^i$ is the measure of agent $i$
- Messaging scheme:
  - Agent 1 computes $A^1_k = \mathbb{E}^1[X|\mathcal{F}^1_k]$. Sends this to agent 2
  - Agent 2 computes $A^2_k = \mathbb{E}^1[X|\mathcal{F}^2_k]$. Sends this to agent 1

Theorem (Teneketzis and Varaiya (1984))
There exists two disjoints subsets $\Omega^A$ and $\Omega^B$, $\Omega = \Omega^A \cup \Omega^B$, such that,

1. For $\omega \in \Omega^A$, after some finite number of iterations, both agents will realize that their priors are inconsistent
2. For $\omega \in \Omega^B$, both agents converge to the same estimate
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2. For \(\omega \in \Omega^B\), both agents converge to the same estimate

- Agents may agree to disagree
- Or they may agree for the wrong reasons!
Inconsistent priors: Example 1

- \( S = \{1, 2, 3\} \), \( \nu^a = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \), \( \nu^b = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \), \( \nu^c = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) \)
- \( \nu^1(S^1, S^2) = \nu^a(S^1) \cdot \nu^b(S^2) \), \( \nu^2(S^1, S^2) = \nu^a(S^1) \cdot \nu^c(S^2) \)
- Assume \( S^1 = 1, S^2 = 3 \). Agent 1 starts first.

Agent 1's possible messages are \( \{1/4, 1/2, 3/4\} \). Sends \( A_1^1 = \mathbb{E}^1[X|S^1] = 1/4 \)

Agent 2 expects \( A_1^1 \in \{1/2, 3/4, 1/2\} \)

Agent 2 cannot interpret \( A_1^1 \)

Declare inconsistency!
Inconsistent priors: Example 2

- $S = \{1, 2, 3\}$, $\nu^a = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\nu^b = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, $\nu^c = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$
- $\nu^1(S^1, S^2) = \nu^a(S^1) \cdot \nu^b(S^2)$, $\nu^2(S^1, S^2) = \nu^a(S^1) \cdot \nu^c(S^2)$
- Assume $S^1 = 3, S^2 = 3$. Agent 1 starts first.

Agent 1's possible messages are $\{1/4, 1/2, 3/4\}$. Sends $A^1_1 = \mathbb{E}^1[X|S^1] = 3/4$

Agent 2 expects $A^1_1 \in \{1/2, 3/4, 1/2\}$

Agent 2 (wrongly) infers that $S^1 = 2$

Agent 2 sends $A^2_1 = \mathbb{E}^2[X|\mathcal{F}^2_1] = 0$

Agent 1 (wrongly) infers that $S^2 = 2$

Agent 1 sends $A^1_2 = \mathbb{E}^2[X|\mathcal{F}^1_2] = 0$

Converges!
Inconsistent priors: Example 2

- $S = \{1, 2, 3\}$, $\nu^a = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\nu^b = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$, $\nu^c = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$
- $\nu^1(S^1, S^2) = \nu^a(S^1) \cdot \nu^b(S^1)$, $\nu^2(S^1, S^2) = \nu^a(S^1) \cdot \nu^c(S^1)$
- Assume $S^1 = 3$, $S^2 = 3$. Agent 1 starts first.

Agent 1's possible messages are $\{1/4, 1/2, 3/4\}$. Sends $A^1_1 = \mathbb{E}^1[X|S^1] = 3/4$

Agent 2 expects $A^1_1 \in \{1/2, 3/4, 1/2\}$

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Converges!
Conclusions and Ongoing Work

REE is an important concept in the study of incomplete markets. But there is not much attention to the REE price discovery process.

- How can agent 1 manipulate its response (conditional expectation) so that the iteration converges to an REE that is more favorable?
- If the virtual bids are repeated, will the supply and demand curves become ‘flatter’ (more elastic)?

How can we study coordination signals that achieve consensus but reveal ‘minimal’ amount of private information?
HAPPY BIRTHDAY DEMOS