EECS 477. HOMEWORK 1 SOLUTIONS.

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1. Summation (20 points)

Prove by mathematical induction that

$$\sum_{i=1}^{n}(2i-1)^2 = \frac{n(4n^2-1)}{3}.$$

Proof. We shall prove it by induction on n

Basis: For $n = 1$ we get $1 = 1$.

Induction step: Suppose that it holds for n. We need to prove the statement for $n + 1$:

$$\sum_{i=1}^{n+1}(2i-1)^2 = \frac{(n+1)(4(n+1)^2-1)}{3}.$$

Let's prove it then:

$$\sum_{i=1}^{n+1}(2i-1)^2 = \sum_{i=1}^{n}(2i-1)^2 + (2n+1)^2$$

By the inductive assumption we have

$$\sum_{i=1}^{n+1}(2i-1)^2 = \frac{n(4n^2-1)}{3} + (2n+1)^2 = \frac{n(4n^2-1)+12n^2+12n+3}{3}.$$

Hence, all we need to show is that

$$n(4n^2-1)+12n^2+12n+3 = (n+1)(4(n+1)^2-1).$$

Expanding both sides establishes this equality.

$\square$

2. Divisibility (20 points)

Prove by mathematical induction that $n^5 - n$ is divisible by 5 for all $n \geq 1$.

Proof. We shall prove it by induction on n

Basis: For $n = 1$ we get 0 which is divisible by 5.

Induction step: Assume the statement holds for n. Let's prove it for $n + 1$: Consider $(n+1)^5 - (n+1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1 = [n^5 - n] + 5(n^4 + 2n^3 + 2n^2 + n)$. The quantity in square brackets is divisible by 5 by the assumption, and the remaining part of the expression is obviously divisible by 5 as well. We are done.

□

3. Quantifiers (10 points)

Is the statement below true or false? Show why.

$$(\exists p \in \mathbf{N})(\exists q \in \mathbf{N})[(p \text{ is prime}) \wedge (q \text{ is prime}) \wedge (pq + 6 \text{ is prime})]$$

Solution: The statement is true: simply take $p = q = 5$. □

4. Who's the killer? (30 points)

A horrendous crime was committed and Inspector Maigret is looking for the killer. His assistants tell him the following:

- If Francois was drunk then either Etienne is the killer or Francois is lying (or both).
- Etienne is the killer or Francois was not drunk and the crime happened after midnight.
- If the crime happened after midnight then either Etienne is the killer or Francois is lying (or both).

Consider the following statements:

A Francois was drunk
B Etienne is the killer
C Francois is lying
D the crime happened after midnight

What follows from the above evidence?

Inspector Maigret knows for sure that sober Francois never lies. What conclusion can he make?

(Model the situation in terms of mathematical notation from Chapter 1 and use it to solve the problem.)

Solution: The three pieces of evidence give us: $(A \Rightarrow (B \vee C)) \wedge (B \vee (\neg A \wedge D)) \wedge (D \Rightarrow (B \vee C))$. This is equivalent to $B \vee (\neg A \wedge D \wedge C)$, which is to say that at least one of the following is true: "Etienne is the killer" or "Francois was sober and lying and the crime happened after midnight". Since the inspector knows that sober Francois never lies, he can make a conclusion that "Etienne is the killer". (A more extensive version would say that $B \wedge (A \vee \neg C)$, that is "Etienne is the killer and either Francois was drunk or not lying"). □

5. Limits (20 points)

a) Find the limit

$$\lim_{n\to\infty} \frac{5^{n+2}}{10^n}.$$

Solution:

$$\frac{5^{n+2}}{10^n} = 25\frac{5^n}{10^n} = 25(1/2)^n.$$

We know that $\lim_{n\to\infty}(1/2)^n = 0$ so the limit of the original expression is zero. □

b) Find the limit

$$\lim_{n\to\infty} \frac{n}{2^n}.$$

Solution: We apply de L'Hopital rule here: $x' = 1$ and $(2^x)' = 2^x \ln 2$ therefore,

$$\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = \frac{1}{\ln 2} \lim_{n\to\infty} (1/2)^n = 0.$$

□